UNIVERSITY OF EDUCATION, WINNEBA

ASSESSING SENIOR HIGH SCHOOL STUDENTS' THINKING LEVELS IN SOLVING PROBLEMS ON CIRCLE THEOREMS: THE CASE OF MFANTSIMAN GIRLS' SENIOR HIGH SCHOOL

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A thesis in the Department of Mathematics Education, Faculty of Science Education, submitted to the school of Graduate Studies in Partial Fulfilment of the requirements for the award of the degree of Master of Philosophy (Mathematics Education) in the University of Education, Winneba.

SEPTEMBER, 2021

DECLARATION

Student's Declaration

I, Victoria Felicia Aidoo-Bervell, declare that this thesis, with the exception of quotations and references contained in published works which have all been identified and duly acknowledged, is entirely my own original work, and it has not been submitted, either in part or whole, for another degree elsewhere.

Signature: …………………………………………….

Date: …………………………………………………

Supervisor's Declaration

I hereby declare that the preparation and presentation of this thesis was supervised in accordance with the guidelines for supervision of the thesis as laid down by the University of Education, Winneba.

Supervisor's Name: Dr. Akayuure Peter

Signature: …………………………………………….

Date: …………………………………………………

DEDICATION

To my parents Bishop Samuel C. Aidoo-Bervell and Late Mrs Sarah Aidoo-Bervell.

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TABLE OF CONTENTS

Content **Page**

LIST OF TABLES

Table **Page**

LIST OF FIGURES

LIST OF ABBREVIATIONS

ICT: Information and Communications Technology

MGSHS: Mfantsiman Girls' Senior High School

NEA: Newman's Error Analysis model

PISA: Programme for International Student Association

SHS: Senior High School

SPSS: Statistical Package for Social Sciences

TIMSS: Trends in International Mathematics and Science Study

WAEC: West African Examination Council

WASSCE: West African Senior High School Certificate Examination

ABSTRACT

Teaching and learning of geometry are to provide students with the ability of critical thinking, problem-solving skills and levels of geometric thinking skills. The researcher after conducting literature studies, noticed that most studies on circle geometry (theorems) consider the interventions of providing effective instructional materials or strategies, and teaching and learning methods which would improve students' proficiency in solving problems on Circle Theorems rather than considering assessing or analysing the cognitive abilities of the students. This study aims to assess SHS students' thinking level using SOLO taxonomy guide in solving problems in the concept of Circle Theorems. A total of 80 SHS 3 students from MGSHS were selected for the study. The study employed the mixed method approach, with sequential explanatory being the design. A cognitive test developed by the researcher in the form of super-item test based on SOLO taxonomy was used as an instrument to collect data from the 80 participants upon which, five were interviewed. Participants' scores were analysed descriptively whiles the interview responses were analysed thematically. The study revealed that most students' level of thinking had attained the highest level; extended abstract levels of SOLO model. The study showed that more students could not explain the reasons in relation to their answers provided to the test items. Students in the study also demonstrated errors such as comprehension, transformation, process skills and encoding errors based on NEA model. The study recommended that mathematics teachers should use SOLO taxonomy as a way of assessing students to monitor their student's solving ability in Circle Theorems.

CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

Mathematics is the science that deals with logic of shape, quantity and arrangement (Elaine, 2014). Mathematics in this study is defined as an important subject of which learners must acquire all the necessary skills, knowledge and understanding that will aid them in the real world and also as an abstract science of quantity (number theory), structure (algebra), space (geometry), and change (mathematical analysis). Mathematics plays a vital role in our lives, especially in the development of a nation and must therefore not to be overlooked. Due to this, Mathematics has been made a compulsory subject to be studied in all fields of Education. Without passing mathematics, it becomes difficult to be able to gain admission into the tertiary (higher) level of education. The general aim of mathematics is to make an individual acquire the mathematical knowledge needed in daily basis to teach, to know how to solve problems, to make them have a method of solving problems and to acquire reasoning methods. For this purpose, to acquire mathematical concepts, one should understand and be able to use the language, symbols and notations of mathematics.

Despite the importance of mathematics, students constantly perform poorly in the subject. Amazigbo (2000) cited in Fabiyi (2017) discoursed that mathematics educators have put in several efforts aimed at identifying the major problems associated with the teaching and learning of mathematics that enhance students' performance in the subject. Some of these problems include poor background of students in mathematics, lack of incentives for teachers, unqualified teachers in the system, lack of learner's interest, students' perception that mathematics is a difficult subject, large class size and the psychological fear of students. In spite of all these noble efforts, the problem of poor achievement in mathematics still pertains in the nation's public examinations.

The chief examiner's report of the West African Senior High School Certificate Examination (WASSCE), identified geometry as one of the branches of mathematics in which students' performance is poor (WAEC, 2019). The 2003 Trends in International Mathematics and Science Study (TIMSS) report by Anamuah-Mensah and Mereku (2005) shows that Ghanaian students scored zero in advanced and higher-level thinking in the content domains tested. Unfortunately, Geometry was one of the areas in which the performance was weak.

Geometry is the branch of mathematics that deals with the study of shapes and their properties, size, positions and dimensions of objects. Geometry has two categories of shapes; the two- dimensional objects also known as plane geometry and the three – dimensional objects also known as solid geometry. For example, points, lines, squares, circles, triangles, polygons are some of the simplest shapes in plane geometry while as cubes, spheres, cylinders are some simple shapes in solid geometry.

Geometry is noticed everywhere around us in our everyday life. Yet somehow, students often do not see these around them or associate them with things in the mathematics classroom. For instance, when students think of angles, they often restrict their thoughts to intersecting lines drawn on paper that can only be measured and constructed by using a protractor and a pair of compasses. But angles are in objects, buildings, hills, trees, waves of the sea, and even in humans (movement of arms and legs). Understanding of geometric concepts is important for representing and solving problems in mathematics and other courses (Herr, 2008). National

Council of Teachers of Mathematics (NCTM, 2003) agreed that geometry is a prerequisite for successive mathematics courses such as trigonometry, mensuration, calculus and advanced algebra. It is also a necessary requirement for science courses such as chemistry and physics; for instance, when solving a problem in chemistry and calculus, the understanding of integrals as the area under a curve in geometry can be very helpful. Architects also use geometry in order to design buildings with interesting shapes and sizes.

Geometry offers a rich source of visualization for the understanding of arithmetical, algebraic and statistical concepts (Battista, 1999 cited in Fabiyi, 2017) and also provides an opportunity for creating spatial understanding and thinking critically. Developing of visualization skills allows students explore mathematical and other problems without the need to produce accurate diagrams or use symbolic representations (Jones, 2002).

Geometry requires students to utilize skills such as measurement, induction, deduction, problem solving, proofs and modelling of real-world experiences. NCTM (2002) stressed the importance of geometry by stating that, "geometry offers an aspect of mathematical thinking that is different from but connected to, the world of numbers". Ozerem (2012) reports "studying geometry is an important component of learning mathematics because it allows students to analyse and interpret the world they live in as well as equip them with tools they can apply in other areas of mathematics". This means one's understanding of the environment we live in, and also been able to apply it in areas of mathematics depends on our understanding of geometry.

Geometry and Algebra are the branches of mathematics, in which students spend the most time learning in their basic and high school mathematics courses which is recognised by NCTM (2003). In Ghana and other countries, geometry is very significant in both the primary and high school's curricula. The teaching syllabus for mathematics in Ghana, Junior High and Senior High schools both stress on the teaching and learning of geometry (CRDD, 2010).

Geometry is connected to every aspect in the mathematics curriculum and to a multitude of situations in real life. Despite the importance of geometry, researchers have revealed many factors that influence the learning of geometry in the classroom, which makes students find it difficult to understand. Some of these factors are: the psychological fear of the topic geometry, inaccessibility of instructional materials and teachers' method of instruction (Fabiyi, 2017), inability of students to deduce theoretical statements into reasoning and also to recognise visually geometrical properties (Laborde, 2005), inappropriate learning of language needed to comprehend and discuss geometrical principles (Swindal, 2000) and issues of extracting information from objects and form both natural and formal concepts by students (Battista, 2009).

Students in the United States are underachieving when compared to other nations in spite of the importance of mathematics and geometry (Mullis et.al, 2000; OECD, 2009; Wilkins & Xin, 2002). The Organization for Economic Co-Operation and Development's (OECD, 2010) 2009 study ranked the mathematics proficiency of 15-year-old students in the United States as 32nd out of 65 countries. In TIMSS 2011, geometry was the aspect of mathematics in which students specifically, United States eighth- graders scored 24 scale score points lower than their overall mathematics average scale score (Mullis, Martin, Foy, & Arora, 2012). This indicates the poor performance of students in the United States eighth-graders in geometry.

Similarly, students in Indonesia mathematics ability especially geometry is low according to the results of PISA (Programme for International Student Association) evaluation in the year 2015. The mean score of mathematical ability of students was 386 while the mean score in PISA was 490 (Ningsih & Paradesa, 2018). Adolphus (2011) also identified that students in Nigeria do not perform well in geometry due to some factors such physical facilities, quality and quantity of teaching staff, attitude of students, parents and government. All these, indicate that students all around the world do not perform well in mathematics especially geometry.

A series of observations in my mathematics classroom has revealed a pervasive problem; the difficulties students have and the errors they make when studying Circle Theorems. Learning and solving Circle Theorems problems are really challenging, but it can be fun if students have a good understanding of the concepts. Therefore, it is necessary to find the appropriate way of teaching and learning that can be used to assist students in understanding the concepts and improve their performance.

The traditional method of teaching has over the years been used to teach Circle Theorems in the mathematics classrooms in Ghana. The traditional method is the teaching approach characterised by lecture or oral exposition. This method is more of teacher-centred than learner-centred. Situations like this, produces students who are able to calculate but do not know how to solve everyday life problems that involve concepts and mathematical skills. This teaching approach has resulted in most senior high school students not able to construct, visualise and justify geometrical concepts (such as Circle Theorems).

With dominance of traditional methods of teaching in mathematics in Ghana combined with students' learning difficulty in solving problems under Circle Theorems, one appropriate approach for improving instruction and students learning could be implementing realistic instructional strategy and taken into consideration students' thinking level in Circle Theorems.

In solving a given problem, a student must be able to select and determine the elements that can be used in problem solving (Mohd Nor & Idris, 2009). The problem-solving process is related to the cognitive domain of the students. Therefore, problem solving is also related to students' mathematics learning achievement. Each problem solving in mathematics has different characteristics, thereby making the response given by each student to the problem solving be different. Biggs and Collis (1982) explain that each stage of cognitive response is the same and increases from simple to abstract. This theory is known as Structure of the Observed Learning Outcome (SOLO).

The SOLO taxonomy is used to classify students' ability to respond to a problem and their performance in assessment. The SOLO taxonomy has five different levels and are hierarchical, i.e., Pre-structural, Uni-structural, Multi-structural, Relational, and Extended abstract. Several studies have shown that applying SOLO taxonomy in learning will help students to study and prepare for the best answer (Lister, Simon, Whalley & Thompson, 2006) and also a student's learning outcome may be understood at any one of the five levels.

Teachers can use SOLO taxonomy to design differentiated learning tasks and to create differentiated success criteria. They can use it with any topic to:

- plan the level of learning required for that topic,
- assess the extent to which each student has reached that level and
- make decisions on next steps for learning.

Several studies have use SOLO taxonomy to analyze students' thinking level comprehensively by using Super-item test model to compile test items. These Super-items are arranged according to the five levels of SOLO taxonomy, starting from the simple question to the more complex one. The Super-items test model is use as an alternate assessment tool for students' cognitive development.

Therefore, it is the aim of this study to use Super-item test based on SOLO taxonomy model to assess students' thinking level in solving problems on Circle Theorems.

1.2 Problem Statement

Recent evidence supports the position that Ghanaian High School students are not learning Geometry in general and Circle Theorem in particular at a deep level, which raises concern to mathematics teachers, parents and the government (Baah-Duodu, Osei-Buabeng, Cornelius, Hegan & Nabie, 2020; Tay & Mensah-Wonkyi, 2018). The chief examiner's annual reports of the West African Examination Council, core mathematics, students performed poorly in mathematics, geometry in particular. Apart from the chief examiners report, results from the end of semester assessment indicated that students' of Mfantsiman Girls' Senior High School have difficulty conceptualizing geometry (Circle Theorem) that is out of 50 students in a class, only 5% of the students passed in the semester examination, which shows massive failure of students.

Geometry as one key component in the study of Mathematics is imperative for students to learn, not only the basics but also to be able to apply them to the real world. Geometry has an important role in the basic and secondary levels. In studying Geometry, students can develop their intuition and spatial visualization abilities. The ability to reason about relationships within and between geometric figures is vital to productive geometric thinking according to National Research Council (NRC, 2005). Students are expected to learn a certain geometric thinking skills when learning geometry to supports further learning of mathematics therefore learning of Geometry should involve a lot of reasoning activities. The nature of geometry as a thinking tool assists in the expansion of logical thinking and reasoning habits and strengthens the interpretation and evaluation of mathematical arguments (Driscoll, Dimatteo, Nikula, & Egan, 2007; McCrone, King, Orihuela & Robinson, 2010). These productive geometric skills can be use in transforming life situations and in problem solving, not only in school but also beyond the classroom in the real world (Wiggins & McTighe, 2008).

Students consider geometry as a difficult topic to learn because of its abstract and complex characteristics especially Circle Theorems; where students have to watch, jot notes and memorise theorems. This often causes cognitive overload and poses negative effect on students' learning. Circle Theorems represent an essential field in the senior high school mathematics curriculum, which makes it a core area in Geometry.

Students are taught the minimal geometrical concepts in the basic level of education, which makes them see geometry as the study of different types of shapes. But since geometry extends beyond the classroom, it encompasses nearly all aspects of life;

therefore, geometry class should be full of deductive and logical thinking. However, students miss out on the necessary deductive and logical thinking skills because geometry is usually not revisited until later in high school making students not mentally prepared for the topic at hand. As a result of these, student's minimal knowledge in geometry, may not proceed to a higher order of geometrical thinking (Jones, 2002).

With this information, one way of measuring what level of students' thinking level in solving Circle Theorems problems is to administer Structure of the Observed Learning Outcomes (SOLO) taxonomy test guide in teaching or classroom instructions.

1.3 Purpose of the Study

The purpose of this study is to assess SHS students' thinking level using SOLO taxonomy guide in solving problems in the concept of Circle Theorems before leaving the school.

1.4 Objectives

In order to achieve the purpose of the study, it is conducted to specifically, focus on the following objectives:

1. To determine, according to SOLO taxonomy SHS students' level of thinking when solving problems on Circle Theorems.
2. To determine how SHS students think in solving problems in Circle Theorems.
3. To examine the errors and misconceptions SHS students' commit in solving problems on Circle Theorems.

4. To determine the relationship between SHS students' demographic variables (age, programme of study and location) and their level of thinking on solving problems on Circle Theorems.

1.5 Research Questions

The research seeks to address the following research questions:

1. What thinking level with respect to SOLO taxonomy, do SHS students attain in solving problems on Circle theorems?
2. How do SHS students think in solving problems in Circle Theorems?
3. What errors and misconceptions do SHS students commit in solving problems on Circle Theorems according to SOLO taxonomy?
4. Do SHS students' demographic variables (age, programme of study and location) influence their level of thinking on solving problems on Circle Theorems?

1.6 Research Hypothesis

Based on research question 4, the following null hypothesis was formulated for the study:

H_0: There is no significance relationship between SHS students' demographic variables (age, programme of study and location) and their thinking level on solving problems on Circle Theorems.

1.7 Significance of the Study

Geometry thinking is very important in the development of mathematical knowledge in students; therefore, the problem of low ability of students to think geometrically must be addressed. One such way of overcoming it is to apply the most suitable method of teaching and learning by considering the thinking level of students.

Knowing the thinking level of students will serve as a guideline for mathematics teachers and mathematics educators, curriculum and professional development planners to arrange learning that is relevant to the level and process of thinking among students so that they do not feel depressed when learning mathematics in general and geometry (Circle Theorems) in particular.

In Ghana, it is hard to find research studies on the use of SOLO taxonomy guide in assessing the cognitive development of students. The result of this study will provide evidence on the significance of using SOLO taxonomy guide to assess SHS students' thinking levels in solving problems on Circle Theorems in the Ghanaian senior high schools. The instrument of this study can also be used as an assessment tool to evaluate the strengths and weaknesses of students' conceptual understanding of Circle Theorems.

From the researcher point of view, the outcome of the study will also determine whether the use of SOLO taxonomy can be used as an alternative and/or supplementary ways of teaching Circle Theorems instead of the teacher talk- and-chalk method of teaching.

1.8 Delimitation of the Study

Delimitations in research are the conditions within the control of the researcher. It also influences the researcher's choices, boundaries and scope of the study. The study was conducted at Saltpond in the Mfantseman West District of the Central Region of Ghana. The study focused on SHS 3 students and only Mfantsiman Girls' Senior High School in the district was used. The main attention was on assessing students' thinking level in solving problems on Circle Theorems based on SOLO taxonomy.

1.9 Organisation of the Study

The study follows the outline described below.

Chapter One: Introduction

In this chapter, the orientation of the study was established and the contextual view of the thesis highlighted. The research questions and significance of the study, as well as the aims and objectives, problem statement, delimitations and definition of terms of the study were addresses.

Chapter Two: Literature Review

The study adopted SOLO taxonomy as the theoretical framework. This framework was used to assess the level of SHS students' thinking in solving problems concerning Circle Theorems. Literature on studies that used the model (SOLO taxonomy) was reviewed together with studies on the nature of geometry and circles and some problems students face in learning geometry. The use of Super-item techniques and some errors and misconceptions on Circle Theorem were also reviewed in this chapter.

Chapter Three: Research Methodology

The study sought to establish the level of thinking attained by students in SHS 3 before writing WASSCE based on SOLO taxonomy. In addressing this problem, the mixed method approach was employed. This consisted of a quantitative data collection, which was based on the cognitive test in the form of super-item test according to SOLO taxonomy, also the qualitative data collection was based on item analysis and interview of students' work. The instrument was subjected to validity and reliability. The study conducted also ensured in ethical manner in this chapter.

Chapter Four: Data Analysis, Results and Discussion

The chapter has discussed the analysis of the performance of the students in the study by looking at the respective questions. The results of the data analysis applied to evaluate the findings of the study and answer the research questions are also presented.

Chapter Five: Summary, Conclusions and Recommendations

This chapter summarises the study and draws a conclusion upon which recommendations are made.

CHAPTER TWO

LITERATURE REVIEW

2.1 Overview

In this chapter, the theoretical framework and learning theory, and also literature review on the nature of geometry in mathematics and Circle theorems in the mathematics curriculum, SOLO taxonomy together with the super-item test, and students' errors and misconceptions based on Newman's error analysis framework have been discussed. A literature review prior to the investigation was carried out. It was done to create a strong base for the research and to find different views about the topic under consideration. The learning framework is based on SOLO taxonomy.

2.2 The Nature of Geometry

The word 'Geometry' comes from two ancient Greek words γεωμετρ.α; geo meaning "earth" and metron meaning "measurement" thus measuring the earth. The origins of geometry are very ancient; probably the oldest branch of mathematics with several ancient cultures developing a form of geometry appropriate for the relationships between lengths, areas and volumes of physical objects. In these ancient times, geometry was used in the measure of land (surveying) and in the construction of religious and cultural artifacts. Examples; the Hindu Vedas, the ancient Egyptian pyramids, Celtic knots and many more.

Euclid, who introduced mathematical rigor and the axiomatic method still in use today, revolutionized geometry. Euclid's book, The Element was arranged into text form the much-accumulated knowledge of geometry around 300 BCE and became the perfect example of the axiomatic-deductive method for centuries. It is known to have been the more widely used, edited or studied and probably no other works apart from the Christian Bible and the Muslim Koran has implemented a greater influence on scientific thinking. The Elements have appeared for more than two millennia, which have dominated all aspects of geometry, and it's teaching (Jones, 2002).

In the time of Euclid there was no clear distinction between physical and geometrical space. The concept of space, since the nineteenth century has experienced a drastic transformation, and raised the question of which geometrical space is the most favorable one for physical space. In the twentieth century, because of the rise of formal mathematics, 'space' (that is either point, line or plane) lost its intuitive content, so today one has to distinguish between physical space, geometrical spaces (in which 'space', point, etc. still have their intuitive meaning) and abstract spaces. Modern geometry considers manifolds, spaces that are considerably more abstract than the familiar Euclidean space, which they only approximately resemble at small scales. These spaces may be endowed with additional structure, which allows one to speak about length. Recently, geometry has many relations to physics as its exemplified by the links between Pseudo-Riemannian geometry and general relativity. One of the youngest -physical theories, string theory, is also very geometric in flavor (Boyer, 1991).

Euclidean geometry was based largely on a set of well-reasoned and highly logical axioms, postulates and deductions in proving propositions or theorems. This postulation approach; even though it has been modified, it is the approach still used in terms of which high school geometry is studied in many countries (Bell in Atebe, 2008). According to Bell and French as cited in Atebe (2008), for over two millennia, Euclidean geometry was the only type of geometry that was studied in high schools in many countries; United Kingdom is among the countries whose mathematics curriculum involved Euclidean geometry until the end of the nineteenth century.

Geometry originated as a practical science concerned with surveys, measurements, areas, and volumes. Among other highlights, notable accomplishments include formulas for lengths, areas and volumes, such as the Pythagorean theorem, circumference and area of a circle, area of a triangle, volume of a cylinder, sphere, and a pyramid. A method of computing certain inaccessible distances or heights based on similarity of geometric figures is attributed to Thales. The development of astronomy led to the emergence of trigonometry and spherical trigonometry, together with the attendant computational techniques (Boyer, 1991).

Geometry has a long history among various branches of Mathematics. It is the study of space and a classification of the way we view the space around us (Leong & Lim-Teo cited in Hoi-Cheung, 2011). In layman terms, geometry describes the relationship between lengths, areas and volumes of physical objects. It is an important component of many aspects of human life from practical measurement and construction (for example in architecture and engineering) to styles of living (Jones, 2002) like designs. Even in the sporting fields, geometry plays a part from how lines are drawn on a sports field to how sportsmen are trained for the best performance and the strategies involved. To our surprise, geometry is also involved in biochemical modeling for medicine and biology. Geometry is undoubtedly more than just shapes and figures on printed books.

Geometry is full of interesting problems and theorems, has many different approaches in teaching, these make the topic interesting to teach in mathematics. Teaching geometry well means to enable learners to succeed in mathematics since geometry has a long history intimately connected to the development of mathematics. Teaching geometry well also involves knowing how to recognize interesting geometrical problems and theorems, appreciating the history and cultural context, and understanding the many and varied uses to which geometry can be put. These tend to make teaching of geometry very demanding and time consuming.

2.3 Importance of Geometry

According to Paulina (2007) cited in Fabiyi (2017) "geometry is a branch of mathematics that deals with the study of different shapes or figures and their properties". Bassarear (2012) also suggested that geometry is the study of shapes and their properties, and studying the relations between and within the figures. Generally, geometry is the study of shapes and their properties, size, positions and dimensions of objects. Geometry has two types of shapes: plane or solid shape, and their properties. The plane shape is a geometrical object with 2-dimensional shapes (length and width/breath), such as circle, square, triangle and so on. The solid shape is a geometrical object with 3-dimensional shapes (length, width and height), such as cone, cylinder, pyramid and so on.

The role of geometry in real life makes it very important in the component of mathematics, as it serves as a prerequisite knowledge to many other areas of mathematics and other subjects such as chemistry, art and also real-life activities in general. Learners can make sense and solve problems in other areas of mathematics and even daily life situations due to the knowledge in geometry. Through the study of geometry, learners are expected to understand and appreciate the beauty of the physical world.

Learning geometry from early age, learners are able to see, know and understand the physical world around them, and continue their education with high level of geometric thinking developing in deductive and inductive systems, as they grow old. NCTM (2003) states that geometry helps students to learn more about geometrical shapes and structures and how to analyze their relationships and characteristics. Daher and Jaber (2010) also stated that geometry helps students build their understanding from informal to more formal thinking and be able to pass from recognizing the different geometric shapes to geometric reasoning and problem solving.

Geometry is in many types of fields and professions in the market today. A number of jobs and occupations demand a working knowledge of geometry to complete the day-to-day requirements. Below are a few examples of professions that need the understanding of geometry in order to complete their daily tasks:

- Computer graphic designers need to know and understand geometry to create realistic 3-dimensional space images. Video game creators and animators are example of graphic designers that need the understanding of how to use and manipulate shapes, which make computing designs easier for computers.
- Medical professionals use geometry to help create a 3-dimensional model of medical issues like tumors in patients. For instance, results from a CT scan can be scaled to a 3D model of that tumor which gives the doctors and surgeons the insight of what issues they need to address for their patient.
- Robotic engineers need to understand geometry in order to know what angles to use for a range of motion by a robot. Having the ability to control these robots down to the smallest movement is all pre-determined by arcs and angles. Some robots are built with a range of vision to detect objects, so measuring out angles and perception are everyday tasks within this profession.
- Fashion designers use geometry on a daily basis to design the perfect look for their customers. Measuring clothing based on body type and angles can make

or break a particular look and feel for someone. As a fashion designer, you will need to know how to take 3D shapes and create patterns for your clients.

- Architectures need the knowledge of geometry to design and construct safe buildings, bridges and roads to minimize dangers and accidents. Engineers and Artisans also need the basic knowledge of the properties of geometry, as well as skills to develop a well-designed building.
- In plumbing the concept of alternate interior angles being equal is used for placement of angles in pipes. In the forestry industries the concept of similar triangles is used in determining heights of trees.
- When describing the location of places or objects or when giving directions, geometric terms such as "adjacent to", "parallel to", and "diagonally from" are used extensively. Geometry is also used as an application in other topics in mathematics and to prepare students to study courses in higher mathematics and sciences

Ramdhani, Usodo and Subanti (2017) identified that various theorems in geometry help learners to develop problem-solving skills, skills of visualization, intuition, deductive reasoning, critical thinking, proof and logical argument. NCTM (2002) stressed the importance of geometry by stating that, "geometry offers an aspect of mathematical thinking that is different from but connected to, the world of numbers". Ozerem (2012) reports that studying geometry allows students to analyse and interpret the world they live in as well as equip them with tools they can apply in other areas of mathematics.

The understanding of the environment we live in, and the ability to do well in areas of mathematics rest on our understanding of geometry. Space, and the shape of things surround us. The very planet we live on is situated in a universe full of interesting stars, surrounded by other planets. All of them have different shapes and sizes. The architectural design used in the northern part of Ghana for example shows a lot of geometric shapes. The roofs of these buildings are in the shape of a cone whiles the houses are cylindrical in shape. Without even thinking about it, we are using those concepts that we learnt in the mathematics class. It is a part of our daily life, whether we watch television or make a drawing. It is part and parcel of who we are.

These are 10 shocking realities of importance of using geometry in our lives:

Spatial understanding

To understand the wonder of the world's shape and appreciate it, one needs to be able to understand and have knowledge of spatial use. In other words, the areas related to space and the position, size and shape of things in it.

Numbers and measurements

When we know how to apply and understand the relationship between shapes and sizes, we will be better prepared to use them in our everyday lives. Geometry provides the knowledge of how to deal with measurements and relationships of lines, angles, surfaces and solids.

Visual Ability

Some people think in shapes and sizes, others think with visual abilities. When visualizing we need the understanding of geometry to be able to do that. Your imagination is like an untapped source of objects that all need to come together in a bigger picture.

In the workplace

Many different scientific and technological fields require knowledge of geometry. Especially in the more advanced and specialized study fields the use and knowledge of Geometry is essential to excelling.

Full use of brain capacity

Geometry helps you to bring together both sides of your brain. In other words, not just be a left-brain thinker, but also a right-brain thinker. The left-brain is the more logical, technical field, whereas the right brain is the part that visualizes and where the artist gets their creative inspiration. Not many people have the ability to make the two connect and work together as one. Geometry will assist in doing that.

Creative use of Geometry

Think of geniuses that created man made wonders. They all made use of geometry to be able to construct and make their creative thinking come to life. Without the use of geometry, it would only have stayed ideas and dreams. The same with architects that design buildings with interesting shapes and sizes. Today architects are very creative in their thinking.

3-D Thinking

Two- and three-Dimensional shapes are originated in geometry. The use of triangles and other shapes strongly influence this. In the fields of television, movies and even little things like puzzles or books all are influenced by geometry.

Preparation

Geometry is a good training ground for students to make use of concrete materials and activities. Those same experiments now will become stepping-stones later in life.

It will prepare you to use many different types of materials and textures together in fluent harmony.

Wider Horizons

By applying geometry students learn to think outside the box. The same solutions can be applied to many other areas of your life. For instance, when studying the different population groups.

Thinking Skills

When you learn to use geometry, you also learn to think logically. This is very important in everyday life, as not everything is easy and understandable. When thinking logically many difficult problems can be made simple, and solutions can be found easily. To reason logically is one thing that you learn in Geometry.

Despite the role of importance geometry plays in our lives, researchers have revealed that many students are not performing well. For instance, students in the United States are underachieving when compared to other nations (Mullis et.al, 2000; OECD, 2009; Wilkins & Xin, 2002). In TIMSS 2011, geometry was the aspect of mathematics in which students specifically, United States eighth- graders scored 24 scale score points lower than their overall mathematics average scale score (Mullis, Martin, Foy, & Arora, 2012). The 2003 Trends in International Mathematics and Science Study (TIMSS) report by Anamuah-Mensah and Mereku (2005) also shows that Ghanaian students scored zero in advanced and higher- level thinking in the content domains tested, in which Geometry was one of the areas candidates' performances was weak.

Similarly, students in Indonesia mathematics ability especially geometry is low according to the results of PISA evaluation in the year 2015 (Ningsih & Paradesa, 2018). Adolphus (2011) also identified that students in Nigeria do not perform well in geometry due to some factors such physical facilities, quality and quantity of teaching staff, attitude of students, parents and government. The chief examiner's report of the West African Examination Council (WAEC) for several years has also identified geometry as one of the branches of mathematics in which students' performance is poorly satisfactory. All these, indicate that students all around the world do not perform well in mathematics especially geometry.

Many researchers have identified some of these reasons as factors that hinders students' difficulties in learning geometry: the psychological fear of the topic geometry, inaccessibility of instructional materials and teachers' method of instruction (Fabiyi, 2017), inability of students to deduce theoretical statements into reasoning and also to recognise visually geometrical properties (Laborde, 2005), inappropriate learning of language needed to comprehend and discuss geometrical principles (Swindal, 2000) and issues of extracting information from objects and form both natural and formal concepts by students (Battista, 2009).

2.4 Reasons why geometry should be included in the mathematics curriculum

Jones (2002) asserted that, studying geometry helps learners to develop skills such as visualization, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argument and proof. In order for the above-mentioned skills to be nurtured and engage learners to think geometrically, the study of geometry should be included in the mathematics curriculum (Jones, 2002). He deliberated further those geometric representations can be used to help learners make sense of other areas of mathematics like fractions and multiplication in Arithmetic; the relationships between the graphs of functions, and graphical representations of data in

Statistics. Other areas of the study such as Science, Arts and Design, Geography and Technology also need spatial reasoning skills in order to work with practical equipment to develop fine motor skills.

Sherard (1981) also established the following seven reasons why geometry is a basic mathematical skill, which should be taught in the senior high school mathematics curriculum:

- Geometry is a basic skill that aids for communication. Our basic speaking and writing vocabularies are rich in many geometric terms, such as point, line, angle, parallel, perpendicular, plane, circle, square, triangle, and rectangle. This geometric terminology helps us to communicate our ideas to others in a precise form.
- Geometry can be applied to many real-life contexts. Measurements around our homes and many other aspects of our daily life activities require geometrical applications.
- Geometry has important applications to many topics in basic mathematics. Many arithmetical, algebraic, and statistical concepts are better understood when given geometric interpretations.
- Geometry provides a valuable mathematical background for further education. In the U.K., for example, Euclidean geometry was a prerequisite for university entrance (French, 2004 cited in Atebe, 2008).
- Geometry forms part of the cultural heritage of humanity. It has an immediate intuitive appeal at a visual level. There are cultural and aesthetic values to be derived from its study. In South Africa, for example, designs in beadwork and many other aspects of ethno-mathematical study make use of a rich collection of geometric terms (Mogari, 2002).

- Geometry, like mathematics, provides a context for developing students' logical reasoning skills (Mogari, 2002; French, 2004 cited in Atebe, 2008).
- Geometry enhances the "development of students' spatial perception and understanding" (NCTM, 2002).

Many studies are also in agreement that the main focus of teaching geometry in schools is to "develop learners' logical thinking abilities" (Atebe, 2008). It is for this reason that geometry is included in the mathematics curriculum; thus, making geometry the central component of school mathematics curriculum in many countries. According to Gonzalez and Herbst as cited in Luneta (2015), geometry is the only subject in high school where students routinely deal with the necessary consequences of abstract properties and held accountable for reading, writing and understanding mathematical proofs. Atebe and Schäfer (2011) also assert that students' general mathematical competencies have been linked closely with their geometric understanding.

The United States of America (USA) Mathematics curriculum lays emphasis on the learners' knowing and being able to apply Mathematics especially geometry (NCTM, 2000). The following objectives were set in the USA to know the importance of geometry in schools:

- Learners should be geometrical literate in order to cope with everyday life.
- Geometry should empower learners to solve problems at workplace.
- Mathematics should prepare the most able learners to be mathematicians, scientist, statistician and engineers.
- Geometry should be viewed and taught as part of culture.

NCTM (2000) also asserted that geometry is taught in the USA from pre-primary to grade 12. The primary school geometry curriculum covers the study of two- and three-dimensional shapes (NCTM, 2000). Reasoning and proof receive the utmost important at all levels. However, reasoning cannot be developed through geometry alone; it should be developed through all disciplines of Mathematics in variety of context at all grades. The USA Mathematics curriculum also stresses on teaching geometry through the inductive approach from primary level through to grade 8 particularly reasoning and proof are to be taught inductively (NCTM, 2000).

Some developed countries such as England and Germany have their own way of teaching geometry. They focus on designing appropriate and relevant procedures for assessment and recording of learners' progress and attainment. Teachers use these new concepts strategies and methodology in a dynamic creative learning environment in the teaching of geometry in order to improve the learners' performance in geometry (Jayaprakash cited by Mamali, 2015).

Also, Japan and China have National Curricular for Mathematics that covers geometry, among other mathematical topics (Japan Society of Mathematics Education, 2000 in Mamali, 2015). Yet, teachers in the two countries have different traditions and ways in which they respond to international development over the years (Jones, Fujita & Ding, 2005). The analysis of lesson given by Mathematics teachers in China and Japan might inform the development of new pedagogical approaches to teaching geometry.

New Jersey Mathematics Curriculum Framework (1997) suggests that for learners to learn properties of geometric figures, they need to deal explicitly with the identification and classification of standard geometric objects by the number of edges and vertices, the number and shapes of the faces, the acuteness of the angles, and so on. This framework suggests exercises that allow learners to (i) cut-and-paste constructions of paper models, (ii) combine shapes to form new shapes and (iii) decompose complex shapes into simpler ones are suitable to promote understanding of geometric properties. This concurs with the perspective of the van Hiele model of the development of geometric thought, where the learner moves in levels from observing and identifying a figure to recognition of its properties, after which the learner understands the interrelationships of the properties of the figures and the axiomatic system within which they are placed (Usiskin, 2003).

The National Council for Curriculum and Assessment (NaCCA) in collaboration with the Ministry of Education (Ministry of Education, 2020) introduced the new standards-based curriculum for Ghanaian basic schools to place learning at the heart of every classroom and to ensure that every learner receives quality education. The curriculum aims at developing individuals to become iterates in mathematics, good problem solvers who are capable of thinking creatively and to be both confidence and competence to participate fully in the Ghanaian society as responsible local and global citizens (Ministry of Education, 2020). Geometry is one of the areas of mathematics that was stressed on in this new curriculum. Based on this, Baah-Duodu, Osei-Buabeng, Ennim, Hegan and Nabie (2020) asserted that the mathematics curriculum should provide ample opportunities for learners to use geometry for practical problem solving through mathematical modeling in both two- and three-dimensions and also develop learners' spatial thinking, visualization and geometrical reasoning.

Royal Society/JMC (2001) stated that the most significant contribution to the improvement in geometry teaching in the United Kingdom (UK) will be made by the development of good models of pedagogy, supported by carefully- designed activities and resources. While the training curriculum specified in the UK for initial training of both primary and secondary teachers contains very little in the way of geometry and how it should be taught, the complexity of the issue means that there is a lack of consensus about what geometry can and should contain within such courses (Royal Society/ JMC, 2001).

The aim of geometry education is not just by learning definitions or theorems or properties of geometrical shapes but also to have the ability of applying these properties in real life problems. To develop spatial reasoning and geometrical thinking in students through the contribution in discussions related to geometry.

A highly-respected British mathematician, Sir Christopher Zeeman quoted that "geometry comprises those branches of mathematics that exploit visual intuition to remember theorems, understand proof, inspire conjecture, perceive reality, and give global insight" (Royal Society/ JMC, 2001). These transferable skills are needed also in other branches of mathematics and science as well. For this reason, Royal Society/ JMC report suggested that the aims of teaching geometry could be summarized as follows:

- To develop spatial awareness, geometrical intuition and the ability to visualize;
- To provide a breadth of geometrical experiences in two and three dimensions;
- To develop knowledge and understanding of and the ability to use geometrical properties and theorems;

- To encourage the development and use of conjecture, deductive reasoning and proof;
- To develop skills of applying geometry through modeling and problem solving in the real-world context;
- To develop useful ICT skills in specifically geometrical contexts;
- To engender a positive attitude to mathematics; and
- To develop an awareness of historical and cultural heritage of geometry in society, and of the contemporary applications of geometry.

There are a lot of reasons why geometry is an important mathematical skill and should be a major part of the learning experience of mathematics at all levels. Geometry serves, among other things, as a unifying theme to the entire mathematics curriculum and as a tool for developing students' skills in logical and deductive reasoning. Geometry provides opportunities for learners to develop spatial awareness, geometrical intuition, and the ability to visualize and use geometrical properties in a variety of real-world contexts (Jones, Fujita & Ding, 2006).

2.5 Teaching and learning geometry

Geometry is one of the longest established branches of mathematics. It has an extensive range of applications and repository historical and cultural background. One of the major achievements of classical geometry was the systematic collection by Euclid of the geometrical knowledge of the ancient Greeks. This has, until comparatively, recently, formed the basis for much of the geometry taught in schools (Royal Society/ JMC, 2001).

Our cultural life such as aesthetic appreciation of art, architecture, music and many other cultural artifacts are visual and therefore involves geometric principles (symmetry perspective, scale, orientation and so on). The ability to interpret visual information is of importance to human existence as we live on a solid plane in 3D world, much of our experience is through visual stimulus. Visual images invite students to understand how the observed images are related to one another and are linked to the fundamental 'building block'. Geometry offers such a way of developing visualization skills (Jones, 2002).

Geometry can be taught effectively, if teachers keep in mind and give some coherence to classroom tasks the following key ideas in geometry and highlight them where appropriate:

1. Invariance: A mathematician Felix Klein in the 1872 revolutionized geometry by defining geometry as the study of the properties of a configuration that are invariant under set of transformations (Jones, 2002). Examples of these invariance propositions are the theorems that states; the angle formed in a semicircle is equal to 90° and the sum of the interior angles of a triangle is equal to 180°. Learners do not always find it straightforward to determine which particular properties are invariant. The use of geometric construction can be very useful in this respect.
2. Symmetry: Symmetry is frequently used to make arguments simpler, and usually more powerful. Symmetry is a key principle in mathematics, particularly geometry. In technical term, symmetry is the transformation of mathematical object, which leaves some properties invariant. For example, the best way of defining quadrilaterals is through their symmetries, except a trapezium, where some theorems in quadrilaterals do not hold.
3. Transformation: Transformation allows learners to develop broad concepts of congruence and similarity and apply them to all figures. Congruent figures are

always related either by a reflection, rotation, slide, or glide reflection. For instance, studying transformation will enable learners to realize that; all parabolas are similar because they can be mapped onto each other, graphs of $y = \sin x$ and $y = \cos x$ are congruent and matrices have powerful geometric applications.

A good knowledge of geometrical facts encourages learners to make logical connections and explain their reasoning. Such facts will develop their spatial thinking and geometrical intuition. Some facts can be introduced informally, others developed deductively or through exploration, therefore a variety of approaches are beneficial in teaching geometry.

At any level, teaching geometry effectively ensure that learners understand the concepts they are learning and the steps that are involved in particular processes rather than solely learning rules. More effective teaching approaches encourage learners to recognize connections between different ways of representing geometric ideas and between geometry and other areas of mathematics. This evidence suggest that it is likely for learners to retain knowledge and skills and enable them to approach new geometrical problems with some confidence through descriptions, demonstrations and justifications.

When planning approaches to teaching and learning geometry, it is important to provide opportunities to learners to investigate spatial ideas and solve real life problems. There is also the need to ensure that there is a good understanding of the basic concepts and language of geometry in order to provide foundations for future work and to enable learners to consider geometrical problems and communicate ideas.

In Ghana, the collection of Adinkra symbols can be used for teaching concepts in geometry such as shapes, congruence and symmetry (i.e., both line and rotational symmetry). The design "Adinkrahene" for instance could be used in the introduction of the concept of circles. This will aid learners to get the opportunity to learn the concept of concentric circles. Also, Ghanaian market women make use of the stretch of arms in the measurement of length, the use of empty tins as a unit measure in the sale of grains, chili among others, indicate that measurement is done well in the Ghanaian society.

Teaching geometry involves knowing how to recognize interesting geometrical problems and theorems, appreciating the history and cultural context of geometry, and understanding the many and varied uses to which geometry is put. This also mean appreciating the full and rich, geometry education can offer to learners when the mathematics curriculum is often dominated by other considerations like the demands of numeracy and algebra.

Geometry teaching makes learners have a habit of critical thinking, precise understanding about geometric concepts and how these concepts relate to each other. In a mathematics classroom, learners can develop critical thinking skills through; (i) their own decision making on how to approach non-procedural geometric problems; (ii) their own choice of the most appropriate ways of geometric representations; (iii) monitoring progress in problem solving and adjustments; (iv) analysis of their own responses to ensure sense making; and, (v) communication of their mathematical ideas effectively such that they connect geometry with their own lives and the wider world. These are some of the ways in which learners are taught to apply critical thinking to their learning.

In teaching geometry, learners should be equipped to consider the criteria for making thoughtful decisions such that they assess the relevance of any rule that they use without simply guessing or applying it. For example, in problems involving parallel lines, rules for alternating, corresponding and co-interior angles should be applied relevantly. In this way learners can make meaning of learning geometry and apply this knowledge to complete shapes where it is relevant.

Concept is an element of understanding and knowledge (*Öksüz* as cited in Alex & Mammen, 2018); therefore, mathematics education should lay emphasis on the teaching of conceptual understanding in geometry. One significant indicator of conceptual understanding is the ability to represent mathematical situations in different ways and knowing how these representations can be useful for different purposes. Conceptual knowledge of geometrical concepts goes further than the skills required to manipulate geometric shapes (Luneta, 2015). According to Cunningham and Roberts as cited in Alex & Mammen (2018), in the process of trying to recall a concept, it is not usually the concept definition that comes to a student's mind but the prior experiences with diagrams, attributes and examples associated with the concept. It is crucial that future teachers know the basic concepts well in order to understand complex concepts.

The acquisition of technical terminology is the key to success in learning geometry. Students need to acquire the correct technical terms and be able to use them correctly to communicate their ideas about concepts in geometry (Atebe & Schäfer, cited by Alex & Mammen, 2018). Sherard (1981) states that our basic speaking and writing vocabularies are rich in many geometric terms, such as point, line, angle, parallel, perpendicular, plane, circle, square, triangle, and rectangle and this geometric terminology helps learners to communicate their ideas to others in precise forms. Geometry is slotted as an important school subject because it provides perspectives for developing students' deductive reasoning abilities and the acquisition of spatial awareness (NCTM, 2003).

2.6 Circles and Circle Theorems (Circle Geometry)

A Circle is a curved two-dimensional shape where all points on the edges are at the same distance from the centre. Circles are everywhere around us, e.g., the surface of a coin, the vehicle tires, dinner plates, car wheels, etc. The most important shape in mathematics is the circle, once some lines are drawn inside circles, it can deduce patterns and theorems that are useful both in a practical and theoretical sense. Circle is a difficult concept to teach and learn for both teachers and students.

The theoretical importance of circles is reflected in the amazing number and variety of situations in science where circles are used to model physical phenomena. Circles are the first approximation to the orbits of planets and of their moons, to the movement of electrons in an atom, to the motion of a vehicle around a curve in the road, and to the shapes of cyclones and galaxies. Spheres and cylinders are the first approximation of the shape of planets and stars, of the trunks of trees, of an exploding fireball, and of a drop of water, and of manufactured objects such as wires, pipes, ball-bearings, balloons, pies and wheels.

Architects use the symmetrical properties of circles when designing athletic tracks, recreational parks, buildings, roundabouts, Ferris-wheels, etc. Artists and painters also find circle almost indispensable in their work. Circular cylinders are used to print newspapers. Engineers exploit the circle's symmetrical properties as seen by the use of the circle in making watches, clocks, bicycles, cars, train, ships, airplanes, radios, telephones, trolleys, etc. civilization has progressed because of the invention of the circular wheels.

Circle geometry is a branch of mathematics that deals with the properties of lines and angles within, on and outside the circle. Circle geometry is a subsection of Euclidean geometry that incorporates the use of theorems, theorem converses, corollaries and axioms. In Ghana, Circle geometry deals with the use of theorems and it is placed under the topic Plane geometry II in the mathematics curriculum, which is to be taught in SHS 2 students (MOE, 2010).

Although, Circle Theorem is one of the important topics in geometry, most students perform poorly due to the difficulties they have when learning. Theorems and properties are the foundation of geometry in general and Circle Theorems in particular. Students learn these theorems and properties by memorizing them rather than exploring and discovering the basics in order to identify which theorem or property best fits in a particular problem when learning Circle Theorems. Due to this, students tend to fail in problems involving these theorems.

There could be several reasons responsible for the poor performance of students in solving problems in Circle Theorems, for example, poor teaching and learning methods employed by teachers, lack of student confidence and motivation and inadequate teaching and learning materials. Other reasons that could contribute to poor performance in Circle Theorem questions are lack of creativity and exploration in the part of students and also theorems of circle geometry problems being intuitively obvious to students. The process of proving theorems often requires students to use results from different sections of geometry and algebra (Stols cited by Chimuka, 2017), which students' may not be able to relate.

Johnston-Wilder & Mason (2005) cited in Tay & Mensah-Wonkyi (2018), and a number of authors have also attributed students' lack of interest and understanding of geometry on teachers' poor teaching skills. They also argued that the everyday primary school tutor has an anxiety of the very word 'geometry.' Therefore, it is difficult to encourage any form of geometry to be taught at all in primary schools, and some books for primary teachers devote little time or space to it. Pupils who proceed to the senior high school, therefore, have very weak foundation in geometry in general and Circle Theorems in particular. Perhaps, one of the reasons why so much time is spent on arithmetic than on geometry in the primary school is that skills and techniques in arithmetic are very much more in evidence.

A study conducted by Kemanka "I" and "Zsoy in Essay sauce (2019) identified that some students are unable to make contact with the concepts of interior, exterior, centre and inscribed angles in circle because they have a confused definitions and properties of the above-mentioned. The researcher further stated that students have difficulty applying the concept of angle properties on different areas such as from triangular region to circles.

According to Oladosu (2014) students' meaning has an effect on both the learner and the facilitator (teacher), which also influence the learning of Circle Theorems. Students' meaning in this study is referred to as the way the student make sense of and understand the concepts they learn. These can be seen in the way students communicate, interpret and represent the concept and how they use the concepts to solve related problems.

Students have a challenge in learning the appropriate language needed for understanding and discussing geometric principles. One of the aims of learning mathematics in Ghana is to understand and be able to use the language, symbols and notation of mathematics (MOE, 2020). Knowing geometric names like circle, triangle, etc. do not mean the students understand their exact meanings or properties and how to even apply them to solve problems. Learners are unable to explain simple terms like segments, chords, sector, among others which gives rise to much confusion in the learning of the theorems.

In learning Circle Theorems, the formal experiences resulting from the regular classroom settings where a more knowledgeable person (teacher) facilitates the learning experiences of the learner in what is to be learned, when to learn it and how it should be learned influence the students' learning. Formal instruction in the classroom may differ depending on the teacher's content knowledge, the teaching situation in the social context, and the teacher's level of thought processes and reflection. Some classroom formal instructional models are learner centered while others are traditional or conventional.

Lack of thorough understanding of Circle Theorems; Circle Theorem is best achieved when students explore the relationships and see the results in a variety of situations. Once students have a clear understanding of these theorems, it becomes mathematically sound but it needs a good teaching approach to deduce these proves. Constructing diagrams in Circle Theorems to solve problems help students to visualize these theorems in various ways, make predictions and confirm their ideas as they explore and prove these theorems.

2.7 Theoretical Framework

Over the years, Van Hiele's model of geometrical thinking and Piaget's theory of cognitive development have been used to assess the thinking ability of learners' geometrical understanding in mathematics.

The Van Hiele's model suggests that learners pass through series of geometrical thinking and, are able to recognize geometric shapes and be able to construct formal geometric proofs (Van Hiele, 1986). A Dutch couple; Pierre and his wife Dina Van Hiele designed the model after experiencing some challenges pertaining to learners' difficulties in learning geometry in their classrooms. Pierre and his wife Dina tried to categorize the geometrical thinking of their students into levels. These are visualization, analysis, abstraction, deduction and rigor. They made the model in such a way that a learner must go through these levels sequentially in order for true understanding of geometry. Thus, a learner cannot be at one level without going through the previous one since each level has its own language and symbols. For instance, the objects to be used in the analysis level are the objects that have been learnt in the visualization stage; this makes the levels to be hierarchical.

Alex and Mammen (2012) cited in Chimuka (2017) asserted that the Van Hiele's model could be applied to any group of learners regardless of their age or gender. Chimuka (2017) further proposed that the model is not a developmental model where learners need to reach a certain age before advancing through the levels, rather the learners' experiences and activities engaged in determines it. For this to happen, the learning environment or instructional strategies to be used must provide experiences that can advance learners from visualization through rigor stages.

Piaget's theory of cognitive development also suggests that children pass through stages of mental development, which involves changes in cognitive process and abilities (Kendra, 2020). These stages of cognitive development were created and developed by a Swiss psychologist, Jean Piaget who was aroused by the curiosity of children upon the reasons they gave for answering questions which involved logical thinking wrongly (Kendra, 2020). These incorrect responses were believed to show the difference between children and adult's thinking.

Children's minds have been for some time now been considered as the smaller versions of adult's minds. Piaget refutes this fact, claiming that the qualitative and quantitative way of thinking differentiate children's minds from adult's minds. Piaget grouped the cognitive development into four stages. These are; Sensory motor (which starts from birth to 2 years), pre-operational (2 to 7 years), concrete operational (7 to 11 years) and formal operational (adolescence to adulthood) stages. Also in Piaget's cognitive development, every child must go through these stages in a continuous transformation of thought. This means the child's biological maturity state and how to interact with the environment develops differently.

Each level of the developmental stages has a different type of intelligence. Even though, the stages are sequential, each child has their own way of progressing through the stages; some might not even reach the later stage. Macleod (2020), "Piaget did not claim that a particular stage was reached at a certain age", descriptions of stages have involved the average age a child must attain at each stage (Macleod, 2020).

As learning progress, one encounters more complex situations, where learning outcomes can be assessed in terms of its quality. Learners understanding of geometry can be explained more clearly, when one goes beyond Van Hiele's model and

Piaget's cognitive stages. A model that can be used in consistent with Van Hiele's ideas to attain this is referred to as SOLO model or taxonomy, which stands for Structure of the Observed Learning Outcome.

2.8 Structured of the observed learning outcome (SOLO) taxonomy / model

John Biggs and Kelvin Collis designed SOLO taxonomy in 1982 as an evidence-based model after researching into samples of learners' thinking in many different areas (Biggs & Collis, 1982). The researchers found out that learners thinking follow a sequence of increasing structural measure of refinement in many different subjects and across different levels (Hook, 2015). The model was created based on Piaget's stages of cognitive development for the cognitive development of learners in school learning context.

SOLO model was first introduced in 2003 to a New Zealand Ministry of Education cluster of primary and secondary schools by Thinking Educational Consultancy to be used in the classroom (Hook, 2015).

The SOLO taxonomy describes new learning outcomes of learners that range from simple and robust form into a deep understanding of subjects. The model makes it possible to identify the level or stage at which a learner is currently operating at, and what needs to be done in order to progress in the cause of teaching, learning and assessing a topic.

The SOLO taxonomy was carefully constructed to analyze learner's responses to assessment tasks (Biggs & Collis, 1982) in some scope and expertise of mathematics including algebra, probability, statistics, geometry, fault analysis and problem-solving (Lim, Wun & Idris, 2010). SOLO taxonomy has been accepted to be used in various studies aside Mathematics: Poetry, History, Geography, Science, Economics and assessing attitudes towards teenage pregnancy (Collis & Davis, 1986; Biggs & Collis, 1982) cited in Hattie & Brown, 2004.

The SOLO taxonomy as an assessment tool is the process of assessing learners' knowledge and skills in answering questions in depth (Biggs & Tang, 2011; Chambers, 2011) cited in Kusmaryono, Suyitno, Dwijanto, & Dwidayati, 2018. Korkmaz & Unsal (2017) agree that the quality and structure of an answer given by a learner to a question can be assessed by SOLO taxonomy. With this, the learners' understanding and thinking in a given problem can be determined.

According to the SOLO taxonomy model, learner's responses can be coded into two aspects; the type of thinking involved and the level of responses given (Lim & Idris, 2006). This shows how learners operate and perform task differently unlike their developmental age, which is consistent as in the sequence of five modes. A mode in the taxonomy is allied with Piaget's cognitive developmental stages (Lim & Idris, 2006). These modes proposed by Pegg, Guit*érrez* and Huerta (1998) are sensori-motor (related with motor activity), ikonic (related with imaging, imagination and language development), concrete symbolic (related with the use and manipulation of written symbols), formal (related with abstract constructs) and lastly, post formal (related with challenging and extending the theoretical constructs developed in the formal mode).

Pegg, Guit*érrez* and Huerta (1998) asserted further that although the fives modes are in common with Piaget's stages of cognitive development; there are at least two main differences. These are:

- When a new mode is evolving, it does not replace or subsume an already existing mode, rather the existing mode tends to change and assist a recent mode developed.

- These recent modes developed also support further growth in an existing mode.

For instance, in a mathematics classroom, a learner's response given in a formal mode can follow a series of concrete symbolic mode today and in a week's time the response given in a concrete symbolic mode can follow a series of formal mode. The question being asked is "was the learner in the concrete symbolic mode or formal mode"? Lim and Idris (2006) stated that there could be a shift in the label from the learner to the response given in a particular task by using SOLO taxonomy. This means that the complexity of the structure of response to a particular task within a mode can be improved by using SOLO taxonomy.

The SOLO taxonomy model categories learners' cognitive abilities in solving problems into five levels, Pre-structural, Uni-structural, Multi-structural, Relational and Extended abstract. These levels are hierarchical according to the learner's cognitive capabilities. Thus, Uni-structural level is higher than pre-structural level (Biggs & Collis, 1982).

The first level of the SOLO model, which is the pre-structural level, is the ignorance stage that exists outside the model i.e., the learning achievement begins before the learning cycle. The next two levels (Uni-structural and multi-structural) are both levels of surface understanding of ideas or facts without integration. The learning achievement is in the cycle of quantitative phase. The last two levels (Relational and Extended abstract) are the deep levels of understanding of ideas and facts with integration. The deep level of understanding does not only characterize integration and connect knowledge but also increase abstraction. The quality of thinking involved in the deep levels is cognitively challenging than the surface levels. This does not mean the deep levels are more difficult (Biggs & Collis, 1982). This put the Relational level in the cycle of qualitative phase whilst the Extended abstract is outside the cycle of qualitative phase.

The five different thinking levels are explained as follows:

Pre-structural level: At this level, learners do not have any knowledge or idea about what they are supposed to learn. They just repeat the question posed to them. They answer questions without logical basis. Learners demonstrate that they do not know the answer to a question.

Uni-structural level: At this level, learners have learned a relevant aspect of the whole. They have limited understanding. They answer questions by focusing only on the information related to the question. Learners have mastered one aspect of the subject matter but do not understand how they come together. This level uses verbs like define, label, draw, name, match, find etc. For example, draw the parts of a circle.

Multi-structural level: At this level, learners are required to remember several relevant aspects of a whole. Learners are able to make some connections but may not understand the relationships very well. They tend to memorize and make greater acquisition of ideas. These ideas are applied or reproduced in a procedural or predetermined manner. Learners' answers questions by focusing on more than one aspect of the question but is not related to each other. Some verbs used under this level are; describe, list, do algorithms, enumerate, combine, etc. For example, list two theorems of circles.

Relational level: At this level, learners can integrate ideas or facts into a whole, recognize relationships and connect ideas to each other. They have understanding of relationships between theory and practice, purposes and significance of ideas. Learners' answers to questions provide explanations that relate relevant details, which often bring concrete facts together. Verbs such as explain causes, compare and contrast, analyze, relate, classify, distinguish, etc. are used. For example, apply the theorem (alternate angles are equal) to find the value of the missing angle.

Extended abstract: At this level, learners can recognize, judge and generalize the whole ideas of their learning in order to use and adapt their knowledge in new situations. They can make connections between their courses, as well as outside world, and use these connections to enhance their understanding. Learners are able to examine the underlying principles and structures behind the ideas they have learned, consider multiple possibilities and refine their academic learning continuously by integrating it with life experience as they engage in the world. Learner's answers to questions go beyond what has been learned, by reasoning forward, anticipating possibilities, making multiple connections, bringing in (or devising) principles to make their knowledge useful in new situations. This level uses verbs such as theorize, generalize, hypothesis, reflect, prove, justify, predict, etc. For example, give a reason why this theorem was used?

SOLO taxonomy provides a practical framework for looking at the learning process and learning outcomes of learners. SOLO taxonomy is aligned with evidence-based practice and effective pedagogies that are useful for teachers.

2.9 The Psychological basis of the levels of SOLO taxonomy

Biggs and Collis (1982) based the SOLO model on the notion that the occurrence of learning yields both qualitative and quantitative outcomes which can be influenced by the teaching procedures and learner's characteristics (Hattie & Brown, 2004). Biggs and Collis further focused the roles each level play on:

- the learner's prior knowledge of the content and relate it to the occurrence of learning,
- the learner's motives and intentions about learning,
- the learner's learning strategies.

As a result, the levels are ordered in terms of various characteristics; from concrete to abstract, an increasing number of organizing dimensions, increasing consistency, and the increasing use of organizing or relating principles.

According to Hattie and Brown (2004), the levels of the SOLO taxonomy have four major ways to increase complexity. These are:

- **Capacity**: Each level of the SOLO model increases the learner's ability to focus and sustain attention and also remember given information. At the surface (Uni-structural and multi-structural) levels, the learner needs only to encode the information given and provide a response by recalling the information. At the deep (Relational and Extended abstract) levels, the learner needs to think beyond more than several relevant things at once, but also how to inter-relate them.
- **Relationship**: Each level of the SOLO model deal with the way in which the questions and answers relate to each other. A Uni-structural level response involves thinking only in terms of one single aspect, which has no possible connections. The Multi-structural level response involves several aspects of thinking, which has no relationship between them. At the Relational level responses are analyzed and appropriate relationships are identified between the many ideas. At the Extended abstract level responses are generalized to

situations not experienced or beyond the given environment.

- **Consistency and Closure**: Consistency and closure are two contradictory parts of the learner. A learner may want to come to a conclusion by answering or closing the question, but at the same time wants to experience consistency so that there is no contradiction between the questions posed, material given, and the answer provided. Instances where there is a greater need for closure, the answer given may have less information, which makes the results less consistent. In contrast, when there is the need for high level of consistency, a learner may come up with an answer that has more information, but may not be able to reach closure if external factors do not permit. For instance, at the Uni-structural level, the learner grasp information at once, but at the Extended abstract level, the learner has to possibly integrate information that are not consistent and must allow inconsistency across contexts.
- **Structure**: At the Uni-structural level, response takes one relevant piece of information to connect both the question and answer. At the Multi-structural level, response takes several relevant pieces of information and connect them to the question but shows no relationship between them. At the Relational level, responses identify and make use of an underlying conceptual structure. At the Extended abstract level, responses require a generalized structure that the learner needs to extend on the original context given.

2.10 Benefits of using SOLO Taxonomy in the classroom

SOLO taxonomy is a powerful approach for teachers to adopt in their classrooms. SOLO taxonomy helps learners to know that the outcomes of their learning is due to the efforts and strategies brought to task by bringing in their own ideas, relating these ideas or extending the ideas to a given task. Learners are able to develop their metacognition, self-regulation, self-efficacy, engagement and resilience through this process.

The levels in SOLO taxonomy helps teachers and learners to give and discuss feedback, feed-forward and feed-up more effectively. This can be done between students – students, students- teachers and teachers- teachers by discussing the level of the task, level of learners' achievement of the task and the next step for learning.

SOLO taxonomy improves learners' understanding of the purpose of everything they do in the classroom if the learning outcomes are made clear, which will enable teachers to reflect and measure the effect on learners' learning outcomes.

SOLO taxonomy is strongly constructed academically, evidence- based and well established for teachers and learners to agree on what level of SOLO the learner has achieved; making SOLO highly reliable.

SOLO taxonomy is a framework to challenge learners to think deeply about loose ideas, connected ideas and extended ideas.

SOLO differentiates each learners learning tasks and learning outcomes. Thus, tasks and outcomes can be at different SOLO levels.

SOLO is used in constructive alignment to design learning intentions and success criteria for outcome- based education.

2.11 Super-item Test

When assessment is made in the SOLO taxonomy, the pre-structural level must not be included in the thinking level since at that level learners usually have no opinion on the topic to be learnt, or the ideas being offered are not relevant (Polter & Kustra, 2012 cited in Korkmaz & Unsal, 2017). This leads to a test item called Super- item test.

Collis, Romberg & Jurdak (1986) built Super-item test based on the SOLO taxonomy as an alternative assessment tool for examining the growth of learners' cognitive ability in solving mathematics problems (Lim, Wun & Idris, 2010). Super-item test consists of a problem situation (stem) and four different questions (items) related to it. The problem situation is often represented by text, diagram or graphic, while the questions (items) represent the four levels of cognitive reasoning defined by SOLO model.

Super-item test model is a test format for learning that ranges from simple questions and increase to the more complex ones (Yulian & Wahyudin, 2019). Learners and the instructor paying attention to the levels of reasoning in the learner do this in the form of solving problems. Firmasari (as cited in Yulian & Wahyudin, 2019) confirmed from his study that the use of Super-item based on SOLO teaching techniques to assess learners' reasoning ability is better than the ordinary teaching materials used to asses learners' achievement in a topic. He further stated that using Super-item techniques enhances learners' ability to understand and reason mathematically.

In this study, super - item test has been used to assess the level of thinking of students in solving problems in Circle Theorems according to the four levels of SOLO taxonomy model, which are uni-structural, multi-structural, Relational and Extended abstract. Each correct response given by the students within any super-item demonstrated their level of thinking based on the SOLO taxonomy.

Super-item test format of item provides more user friendly and effective way to determine the learners' ability level and discovers their strengths and weakness if they are not succeeding past a certain level.

2.12 Conceptual Framework

Conceptual framework is the description of the main independent and dependent variables of the study and relationship among them. Independent variables are conditions or characteristics that are manipulated to ascertain the relationship to observe phenomenon and dependent variables are conditions that appear to change as the independent variable changes (Orodho, 2005). In this study the independent variable is the use of the four types of the SOLO taxonomy and the dependent variable is the level of thinking of the SHS 3 students. The four types of the SOLO taxonomy that was used in this study are namely; Uni-structural, Multi-structural, Relational and Extended abstract levels.

- **Uni-structural**. Students focus on only one relevant information or idea to get solution. For example, the student might use and refer to the picture given in the stem and try to name the parts of a circle.
- **Multi-structural**. Students use the more relevant ideas to respond to the question given, but they are not able to integrate it. For example, students are able to describe the parts of a circle.
- **Relational**. Students are able to integrate each aspect of the given information into a systematic structure. The information provided is enough to solve the problem. For example, students are able to relate the right theorem to a given question.
- **Extended Abstract**. Students are able to generalize the structure into a new and more abstract concept. For example, the student may have the competence

to generalize, analyze or draw conclusions to their procedure of working.

These levels are hierarchical according to the learner's cognitive capabilities as shown in Figure 1.

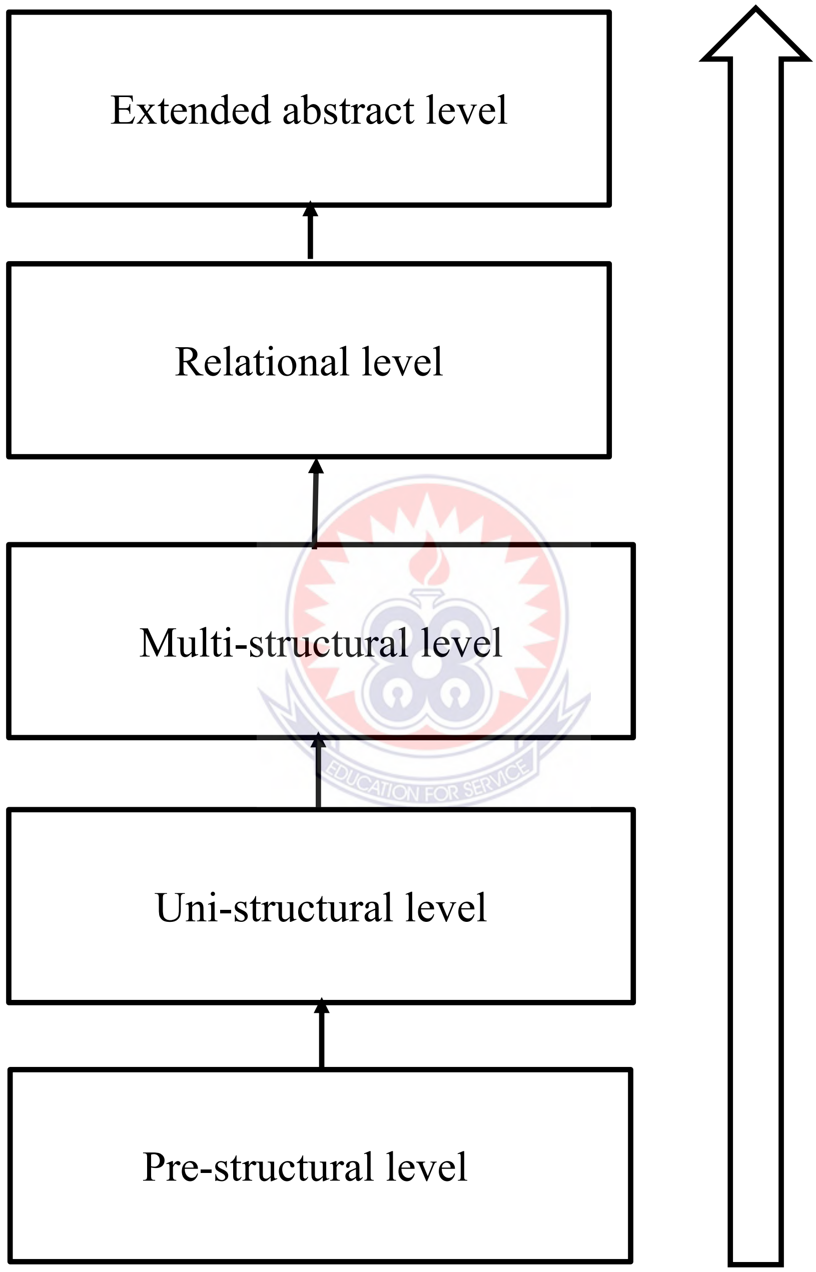


Figure 1: Levels of the SOLO taxonomy from the lowest to the highest

Based on the description above, it can be concluded that the test items to be used in the study in the form of super-item test model is good for assessing the cognitive levels of students. The test items in these super-item tests are set from the simplest task to the complex task (Lim, Wun & Idris, 2010).

2.13 Learners' Errors and Misconceptions

In solving mathematical problems, students must follow several processes to attain their final answer. Polya (1973) cited in Abdul, Nur Liyana & Malina (2015) has four phases in solving a mathematical problem: a) understand the problem, b) devise a strategy, c) carry out the plan and d) examine the solution. With the help of these processes, students can solve mathematical problems with ease. However, not all students are able to solve mathematical problems because of the different levels of thinking and difficulties in understanding instructional strategies. This causes students to make various errors and misconceptions.

An error is the deviation from what is known to be right. Luneta (2008) as cited in Luneta (2015) defined error as 'simple symptoms of the difficulties a student is encountering during a learning experience'. Mutara (2015) cited in Awuah (2018) also defined errors as an evident gotten from something not quite right after using an inappropriate means to arrive at the solution. For instance, in solving an equation $2x = 4$, a student may find the value of x by dividing both sides of the equation by 2 to get the answer to be $x = 2$. However, if a student decides to subtract 2 from both sides of the equation and also arrive at $x = 2$, then the student have committed an error since she used an inappropriate means to arrive at the answer even though the answer is correct.

Errors are classified into two groups, namely systematic and non-systematic errors. Systematic error is an error that occurs intentionally due to the repetitions of wrong answers. Systematic errors are difficult to detect for a long period of time since the answers given to a particular problem may be correct countlessly. These errors become permanent and cannot be easily corrected by learners themselves unless learners are assisted to become aware of them. Non-systematic error is an unintentional and non-repetitions of errors that occur especially in calculations, which result in wrong answers (Awuah, 2018). These errors normally happen as a result of carelessness and are easily corrected by learners themselves.

Misconceptions are systematic errors that arise in students' learning. These errors are difficult to be identified since they happen unconsciously by students. When there is a drawback related to any topic due to conceptions such as beliefs, theories, explanations and meaning in the students' mind, then misconceptions have occurred (Osborne & Wittrock, 1983 cited in Essay sauce, 2019). For instance, students normally consider the diagrams in Figure 2 and Figure 3 to be the same and therefore apply the same theorem in solving them. But in Figure 2, the angle at the circumference does not form twice the angle at the centre inside the cyclic quadrilateral but at the outside. The students need to understand that angles must be subtended from the same arc. This gives to a misconception of representatives and if this misconception is not corrected errors set in.

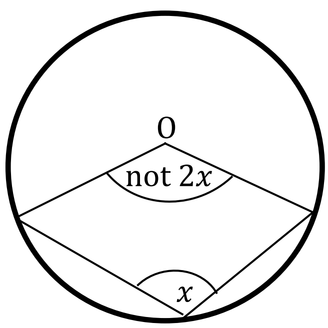


Figure 2: Misconception in theorem

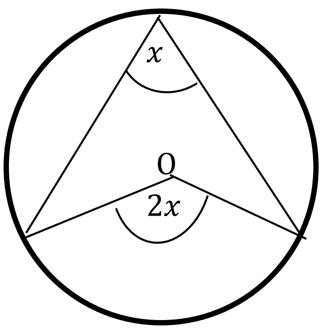


Figure 3: The right theorem

The concept of circle theorem has been identified as being more difficult to understand because of its abstract nature. This has given rise to comprehensive research on some errors and misconceptions of students on circle geometry (Ubuz, 1999 cited in Essay Sauce, 2019; Ozerem, 2012). Ubuz (1999) investigated on grade 10 and grade 11 students' understanding of angle concept in geometry according to their errors, misconceptions and gender in Ankara. The participants who undertook in the research study were required to take a test consisting of 11 open-ended questions. The study revealed that: there was an increase in achievement level of the students; students did not know the meaning of a triangle and its properties; students focused on the geometric figures instead of their properties to answer questions in the test and lastly, most female students gave incorrect answers although they were more successful whilst the male students avoided answering questions, they had no idea.

Ozerem (2012) also conducted a study in identifying the misconceptions that arise during the learning process of geometry among 7th grade college students at Turk Maarif Koleji in Cyrus. The study used the descriptive methodology and student interview to analyze and interpret the results. The study revealed that: 7th grade students have some misconceptions due to lack of knowledge in the topic geometry; reasoning and basic operation mistakes were committed; students were unable to describe some concepts and remember formula related to geometry, and students got confused at recognizing the shapes in geometry due to human perception.

The errors students make are associated with the inability to concentrate while solving problems, lacked reasoning skills, memory overload and fail to identify certain important features in a problem (Higgins, Ryan, Swam & Williams, 2002 in Awuah, 2018). This means that students are able to understand a topic being taught but tend to forget the procedures involved few months after and make errors while solving problems related to it.

Various research studies from different theoretical perspectives have shown that learners have perceptions that hinder their learning of circle geometry concepts and create misconceptions (Ozerem, 2012). This has led to numerous mathematical researchers and educators to carry out error analysis, i.e., the process of studying and analyzing learner's work with a view of finding explanations for what has caused them to improve learners' performance. Some of these errors are highlighted here.

Riccomini (2005) cited in Awuah (2018), classified errors into two; namely: systematic errors and unsystematic errors. He defined systematic error as an error made intentionally as a result of poor reasoning; these errors reoccur most times due to the learner's perception that the answers given to a particular question is correct. This indicates that the learners had the question wrong because of a misconception created on that topic. These misconceptions need to be made known to the learners for corrections. On the other hand, he defined unsystematic error as an error committed unintentionally; they do not reoccur and learners can correct them easily by themselves (Awuah, 2018).

Cheng –Fei (2012) also categorized errors student's make as procedural, factual and conceptual. He defined procedural error as the failure for a student to follow the right step or procedural in solving mathematical problems, factual error is defined as the error student's make due to lack of mastery of a basic fact or the inability to recall a certain fact and lastly, conceptual error is made as a result of poor understanding of a specific concept in mathematics. He asserted further that these errors occur because of lack of knowledge, poor attention and carelessness, curriculum materials and instructional design and method of delivery in the mathematics classroom. Therefore, he recommended that teachers are to identify the errors and misconceptions students' make frequently for proper application of instructions and provision of remedy.

According to Newman (1977) as cited in Fitriani, Turmudi & Prabawanto (2018), he classified errors into five groups; namely: reading error, comprehension error, transformation error, process skills error and encoding error. These errors can be analyzed by using Newman's Error Analysis (NEA). NEA is a framework used to identify and analyze students' errors in solving mathematical problems.

Based on Newman's Error Analysis, reading error is the inability for learners to read the given problem and figure out the words or symbols given in the questions. Comprehension error is the second type of error for which learners are unable to understand or relate to the symbols, expressions and problems given in the question after a thorough reading. Transformation error, which is the third type of error, is the inability of learners to choose the appropriate formula or method or property and relating it to solve a given problem. The fourth type of error is the process skill error, it deals with the inability of learners to use the correct method or operations or make a mistake in the procedures. Encoding error is the last type, it looks into the mistakes of students in generating and justifying or drawing conclusions of the answers they had given.

A study conducted by Trance (2013) cited in Abdul, Nur & Marlina (2015), found out that 70% of the participants in their study made errors in Comprehension and Transformation based on NEA. The study's aim was to assess the progress and performance of students in the concept of Algebra. Participants were made to write an examination based on NEA and the questions demanded participants to complete the problems related to Algebra orally before showing the processes of calculation.

Because of the massive failure of students on the errors, the study recommended that students should be given extra tuition on errors such as comprehension and transformation before solving problems on Algebra to minimize these errors.

Errors are indication of misconceptions students acquire in their working processes. Therefore, it is important for students to be made aware of the errors they make when solving mathematical problems. This will help the students to reflect and correct these errors to avoid repetition. Teachers are also required to pay particular attention to the errors students make in the mathematics classroom and to provide the right remedy or choose the appropriate strategy or model or learning media that will help the students to understand and improve their problem- solving skills to reduce errors in the learning of mathematics (Fitriani, Turmudi & Prabawanto, 2018).

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Overview

This chapter discusses to the diverse techniques, methods and procedures that are used in the process of employing research design or research methods (Creswell, 2009; 2010). This chapter outlines how this study was carried out. It includes a presentation of research design and its justification, qualitative and quantitative methods, research population and sample, data collection procedures; research instruments, validity and reliability issues and lastly, ethical considerations. It also provides a description of the research design and subsequent methods of data collection.

3.2 Research Design

According to Creswell (2009), research design is a set of procedures and tools to be surveyed in seeking answer to the research problem. The research design defines the model, which is used in the study. This study adopted the mixed method research approach. A mixed method research is the type of research in which the researcher syndicates essentials of qualitative and quantitative research approaches for the purposes of study of thoughtful and justification. The method was convenient in this study because it offset the weaknesses of both qualitative and quantitative research methods. This is due to the fact that, in a situation where the quantitative research method cannot be used to interpret context or setting in which people behave, the qualitative method is use. The quantitative method makes it easy to generalize findings to a large group.

Using mixed method allows for additional information such as terms, pictures and narratives to numbers and also utilises numbers to add precise quantitative data to words, pictures and narratives. The mixed method allows the researcher to deal with a wider and richer range of research questions and not pertain to only one particular method of research. In using mixed method, data collection and analysis occur sequentially. The sequential explanatory design was used for the study, since quantitative data was first collected and analysed followed by a qualitative data analysis for further explanations on the quantitative results.

3.3 Population of the Study

The population for this study was all 2020-2021 academic year SHS three students in Mfantsiman Girls' Senior High School (MGSHS) in Mfantseman West District in Central Region of Ghana. MGSHS students were used because students have been underperforming when it comes to the concept of Circle Theorems for several years. It is my belief that students' cognitive abilities have not been considered when teaching the topic Circle Theorems. Hence, MGSHS was chosen for the study.

3.4 Sample and Sampling Procedure

The sample size for this study comprised a total of eight (80) students with ages ranging from 16 to 20 years. Majority of the students aged between 16 and 18 years, which had a percentage of 82.5. The wide age range of students did not pose a problem within the study as the tasks given were for assessing the students' cognitive abilities. The study employed purposive sampling to select three (3) third year classes according to their programme of study (General Science, General Art and Business) to respond to the Super-item test questions based on SOLO taxonomy. The sample size distribution is shown in Table 1.

Table 1: Sample size distribution

	Programme of study			Age					Locality	
	General Science	General Arts	Business	16	17	18	19	20	Urban	Rural
Number of Students	40	15	25	13	40	20	6	1	54	26
Total	80			80					80	

Table 1 shows the breakdown of the number of students (population) in MGSHS used for the study. This consists of forty (40) General Science students, fifteen (15) General Arts students and twenty-five (25) Business students. Also, fifty-four (54) students come from the urban area and twenty-six (26) from the rural area, with ages of students; thirteen (13) 16 years students, forty (40) students 17 years students, twenty (20) 18 years students, six (6) 19 years students and one (1) 20 years student.

3.5 Data Collection Instrument and Method

The instruments used in this study was mainly cognitive test and an unstructured interview. The test consisted of five (5) questions designed according to SOLO Super-item test format. Each Super-item consisted of four items. The items represented the four levels of reasoning defined by SOLO (Uni-structural, Multi-structural, Relational and Extended abstract). This was done to help measure the cognitive skills of learners in a variety of ways. There were no multiple-choice questions. This was to minimize or eliminate guessing and also to avoid awarding marks to learners on aspects they do not understand. The study used cognitive test to allow learners to express themselves freely without fear or shyness. Cognitive test identifies the strengths and weakness of learners in solving a problem. Participants completed demographic information on their age, locality and the programme of study. This information was entered on the front page of the question paper of the test. Teachers in the mathematics department were briefed to assist in the administering of the test to ensure that the data collected from the participants were reliable.

The process on how students answered the test items given to them was, however, gathered through an unstructured interview as a follow-up. The interview was also conducted to identify some misconceptions students had when solving questions on Circle Theorems. The interview consisted of five (5) out of eighty (80) students of mixed ability, which was based on their written response to the test items. A self-constructed super-item test was used to collect data. The item is structured as follows.

3.5.1 The super-item

The super-item test was constructed to ascertain students' level of thinking of solving problems on Circle Theorems based on SOLO Taxonomy. According to Yulian (2019), given students an appropriate learning model will enhance their learning achievements or outcomes. A good learning outcome indicates if the instructional objectives prepared were achieved. One such criterion is to construct test items from simple to abstract. This is necessary because test items serve as the basis for written assessments of mental attributes, and the concepts they express must be stated in brief and clear manner. Being able to draw true and accurate inferences from a test's scores is highly dependent on paying attention to the way the test's items or exercises are constructed (Osterlind, 1998). To make relevant inferences about the mental characteristics a person taking an examination or test, the test items must represent a particular psychological framework or domain of a material. Without a clear correlation between a test item and a psychological concept or a domain of a material, the test item becomes meaningless and purposeless. The interpretability of a test's scores is directly related to the item and exercise efficiency. Concurrent with the concept of score interpretability is the idea that using only carefully designed objects on a test is the primary method by which a professional test developer eliminates unnecessary error variance, or measurement errors, and thereby increases the reliability of a test score (Osterlind, 1998).

The key action verbs prescribed by SOLO taxonomy were used in the construction of the items. For example; *name, identify* and *find* were used to draw out the uni-structural level of SOLO taxonomy (see columns 2 and 3 of Table 2). Each content area was taken from the mathematics (core) syllabus based on the scope of Circle Theorems (Ministry of Education, 2012). An item specification table was designed based on the scope of the SOLO taxonomy used in the SHS mathematics (core) curriculum. The entire Super-item test was made up of five items (see Appendix C) with each item covering the top four levels of the SOLO taxonomy. In Table 2, the third item of the super-item test (see Figure 4) is used to illustrate the SOLO levels, the level competencies in Circle Theorems and exemplars of the level competencies.

3. The figure shows a circular floor which has points P, Q, R and S on the circumference. If it is given that $\angle PQR = 54^0$ and $\angle STR = 76°$, use the information to answer the following:

   i) Define the theorem that will be used to find the angle at S

   ii) Calculate the value of α.

   iii) What is the relationship between the angles at Q and at S?

   iv) Give reason for your solution

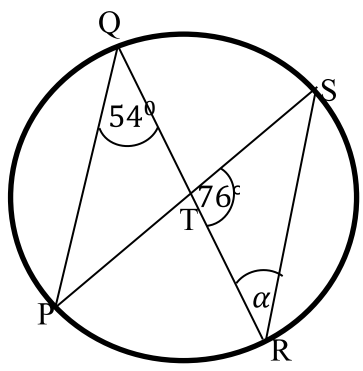


Figure 4 Item 3 of the super-item

Table 2: SOLO levels, level competencies in Circle theorems and exemplars

SOLO level	Level competencies	Exemplars of level competencies
I. Pre-structural level	Learner does not have any knowledge of the concept(s).	• No evidence of working.
II. Uni-structural	Learner recognizes unrelated isolated parts of the concept(s).	• Recognizing the triangle and/or angle the question is about (i.e., ΔRTS and angle $\angle TRS$)
III. Multi-structural	Learner is able to integrate a related part of the concept(s) to see another concept.	• Stating that the sum of angles in ΔRTS is 180° (i.e., $\angle RST + \angle STR + \angle TRS = 180°$); • Replacing $\angle STR$ and $\angle TRS$ in the equation, i.e., $\angle RST + 76^o + \alpha = 180°$ ……………………….. (*equ 1*)
IV. Relational	Learner is able to integrate related parts of the concept(s) into a systematic structure (or have an abstract concept of it).	• Recognizing that $\angle PQT$ *and* $\angle RST$ are equal because of the theorem (angles subtended by the same arc or chord in the same segment are equal), and hence, substituting fully into the equation $54^o + 76^o + \alpha = 180°$ …… (*equ 2*)
V. Extended Abstract	Learner is able to use the generalized structure to solve problems in new situations and/or formulate extended structures.	• Solving the equation: $130^o + \alpha = 180°$; $\Longrightarrow \alpha = 50°$. • Checking back; substituting results into equation 2, i.e., $54^o + 76^o + \alpha = 180°$ to see if the results are true, i.e., $54^o + 76^o + 50 = 180°$.

See Appendix B for the items specification grid that guided the construction of the Circle theorems questions using the SOLO taxonomy. In the grid, the items at the uni-structural level and each item were given a score weight of 1 mark except item 2, which had 2 marks. For multi-structural level, five (5) items with scores of two (2) marks assigned to each test item with the exception of items 1 and 2, which had 1 mark each. The items on the multi-structural level used verbs such as state and calculate to determine if the students have conceptual understanding of Circle theorems and use it to solve for the missing angles. Students were asked to compare and contrast two given diagrams in the relational level. Compare and contrast were the key verb used, and 4 marks was given to item 1 and 1mark given to item 2 and 3 whilst, two items 4 and 5, 2 marks each was assigned to them. This question was tailored on knowing the process the students used in solving the items. Lastly on Table 2 is the extended abstract level where students were asked to draw conclusions and make inductive reasoning of the items involved. Three (3) items were asked in this level in the test and two marks were assigned to each of the items.

3.5.2 Unstructured Interview

The study also used unstructured interview to enable the researcher identify the causes of the errors and misconceptions committed by the students in solving problems on Circle theorems. According to Alshenqeeti (2014), structured interviews are interviews mostly used in quantitative research. He further stated that, this form of interview has questions planned by the interviewer, and it mostly has response of 'yes' or 'no'. Such interviews give very little freedom to both the interviewer and the interviewee. However, unlike the structured interview, unstructured interviews have an open situation where a greater flexibility and freedom is offered to both the interviewer and the interviewee in the contest of planning, organizing and implementing the content of the interview questions (Gubrium & Holstein, 2002; p.35) as cited by Alshenqeeti (2014). In this study, the focus was on some errors and misconceptions committed by students in solving problems on Circle theorems. Open ended questions (unstructured interviews) were directed to five students on their response to the test items to give clarity on the causes of these errors and misconceptions.

3.6 Validity and Reliability of the Study

This section describes the validity and reliability of the study. Validity refers to the extent to which interpretations are made based on how numerical scores are suitable, expressive and convenient to the sample (McMillan & Schumacher, 2001). Validity also determines whether the instruments used in a study provides an acceptable sample of items that represent a concept (De Vos et al., 2005). In this study, both construct and content validity were examined to check if the test questions really measured the concepts that I assumed it measured.

Content validity requires an instrument to adequately cover every form of the content that it should with the variables to be measured. In other words, the instrument should cover the entire domain related to the variable it is intended to measure. Content validity answers the question of how well an assessment tool measures what it is supposed to measure.

Construct validity gives evidence on the relationship between the content of the instrument and the construct it is supposed to measure. Construct validity determines whether the test results are related to what they ought to be related to, and what they ought not to relate to. The super-items were framed to examine the thinking level in relation to Circle Theorems, which is a core content area in the SHS mathematics curriculum. The curriculum covers Circle Theorem one to theorem seven, one question on these theorems is usually set for WASSCE.

Reliability of a measurement process is the constancy or evenness of the measurement. This means that if the same variable is measured under the same conditions, a numerical result each time it is applied; it does not vary unless there are variations in the variable being measured (De Vos, 2005). The reliability of a test or instrument refers to the extent to which it consistently measures what it is supposed to measure. A test is reliable to the level that it measures precisely and dependably, resilient similar results when administered a number of times (Creswell, 2010). To ensure reliability of the data collected in this study, the contents of the test went through corroboration from an autonomous body (3 mathematics experts and my supervisor) knowledgeable in Mathematics education to establish the level to which the contents of the test items were in synchronization with the intended purpose as well as the SHS mathematics curriculum. Any suggestions and input from the corroboration activity from the professional colleagues and supervisors, led to reframing, addition and deletion some existing items. For example, question 2(ii) was reconstructed to "State the theorem used to solve for the angle" (see page 59) to reflect thinking levels in solving problems in Circle Theorems.

3.7 Data Processing

The scripts of participants were marked out of 30 marks using the scoring scheme shown in the last column of the item specification Table (see Table 2). The scores for each of the four items for each question of every participant were tabulated in matrix form and the totals computed for each participant. The data were coded and keyed into SPSS software. The variables included location (1-urban, 2-rural), programme of study (1-General Science, 2-General Arts, 3-Business) and age (1-16, 2-17, 3-18, 4-19, 5-20), and scores rendered to each item. Marked scripts were then categorized into the following groupings to enable the researcher answer the research questions.

a) Grouping according to the SOLO taxonomy levels
b) Grouping according to how students think based on the theorems
c) Grouping according to the errors students made
d) Grouping according to the demographic variables (age, location and programme of study)

3.8 Data Analysis Procedure

Data analysis is the process of analysing collected data to a controllable knowledge, developing patterns and performing statistical analyses (Tavakoli, 2012). Two types of data were analysed that is qualitative data and quantitative data.

3.8.1 Qualitative Data Analysis

Qualitative data was analysed by the thematic analysis. The students' answers were subjected to thematic areas to explore their thinking level based on the student's performance regarding errors and misconceptions at the different cognitive levels on the SOLO taxonomy and an unstructured interview session was also conducted to gain better explanations and identify the causes of some misconceptions on the student's response to the test items.

3.8.2 Quantitative Data Analysis

Quantitative data was analysed by using descriptive statistics to provide answers to the research questions. Descriptive statistics involved calculating the frequencies of the students' outcome in relation to the levels of SOLO taxonomy. Descriptive statistics were used to compare the outcome of the students based on SOLO taxonomy and also to inform the researcher of the level of the SOLO taxonomy where students achieved the most or lacked knowledge.

To determine how many students attained and remained at each specific thinking level in circle theorems based on SOLO taxonomy, the scores of the participants were computed for each thinking level of the items and graded using the modified grading system similar to Van Hieles (1986). Table 3 shows how the grading was done.

Table 3: Grading system of the scores of the participants

Level	Items for each category computed	Highest possible score	Minimum score to attain a category
Uni-structural	Q1I+Q2I+Q3I+Q4I+Q5I	6	≥ 3
Multi-structural	Q1II+Q2II+Q3II+Q4II+Q5II	8	≥ 4
Relational	Q1III+Q2III+Q3III+Q4III+Q5III	10	≥ 5
Extended Abstract	Q1IV+Q3IV+Q4IV	6	≥ 3

With the above grading system, if a participant scores below 3 after summing all four scores for Uni-structural level, she is categorised as not attaining that level. If her total score is greater than or equal to 3, the participant is said to have attained Uni-structural level. For Multi-structural, the highest possible score for all four items is 8. Therefore, a participant is placed at that level, if she attains Uni-structural level and scores a total score of 4 or higher. For Relational level, the participant must first attain multi-structural level and obtain a total score of 5 or higher. Finally, for extended abstract level, the participant must first attain Relational level and obtain a total score of 3 or higher. The essence for this grading system is to ensure that no one is categorised in two levels and identify those who are thinking at a higher level, a Relational thinker for example can also think at the multi-structural level.

A t-test was conducted to compare if there is a significant difference between students' level of thinking and their locality (urban and rural) and one-way ANOVA was conducted to compare if there is a significant difference between students' level of thinking and their age and programme of study.

3.9 Research Ethics

Ethics is defined as a method, procedure or perspective for deciding how to act and for analyzing complex problems and issues (Rensnik, 2011). Ethical issues require researchers to avoid placing participants in a situation where they might be at risk or harmed either physically or psychologically as a result of their participation in a study. On this basis these ethical issues were observed to ensure the study was voids of any unethical issues.

Participants of the study remained anonymous. All the participants were handled with respect. Each one was given orientation, which explained the purpose of the study and their rights such as withdrawal from participation should they wish to do so without being forced to give an explanation.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.0 Overview

The preceding chapter presented methods of data collection and measures taken to ensure rigor in this study. This chapter presents the results and discussions of the data analyzed. The main focus of this study was to use SOLO taxonomy to assess SHS students' thinking level in solving problems in Circle Theorems. Three research questions were raised:

1. What thinking level with respect to SOLO taxonomy, do SHS students attain in solving problems on Circle theorems?
2. How do SHS students think in solving problems in Circle Theorems?
3. What errors and misconceptions do SHS students commit when solving problems on Circle Theorems according to SOLO taxonomy?
4. Do SHS students' demographic variables (age, programme of study and location) influence their level of thinking on solving problems on Circle Theorems?

To answer these questions, qualitative and quantitative data analysis were used to analyze the data for clear generalizations. In analyzing the data in this study, the researcher organized it into three components, the first component showed the results of the general performance of the learners in relation to each question items, the second component presented the errors and misconception rate of the participants during their response to the question items and the final component reported on relationship between the four levels of SOLO taxonomy and participants programme of study and age group.

In this study, student's scores were measured based on the four levels of SOLO taxonomy and also in line with the concept of Circle Theorems. In all there were five questions with each having four items based on the following: Uni-structural, Multi-structural, Relational and Extended abstract for students to respond to them. In the qualitative analysis, students' performance regarding errors and misconceptions at the different cognitive levels based on SOLO taxonomy and an unstructured interview session were looked at. When students are given a question to solve, they are unable to arrive at a desirable conclusion based on certain errors committed and misconceptions they had. This study also explored the Newman's error analysis to analyze students' errors and misconceptions.

4.1 Research question 1: What thinking level with respect to SOLO taxonomy do SHS students attain in solving problems on Circle Theorems?

This question was answered by using the scores of the students in the cognitive test on Circle Theorems based on the SOLO taxonomy levels of thinking. Table 4 presents the general performance of students in the five test items based on the super-item.

Table 4: Total scores obtained by Participants in the SOLO super-item test

Scores	Number of participants (n)	Cumulative (n)	% (n)	Cumulative %
1-5	5	5	6.25	6.25
6-10	12	17	15.00	21.25
11-15	14	31	17.50	38.75
16-20	22	53	27.50	66.25
21-25	17	70	21.25	87.50
26-30	10	80	12.50	100.00

The results in Table 4 shows that 38.75% (n=31) of the students obtained half and less of the total score, while 61.25% (n = 49) obtained more than half of the total scores allocated to the test. Interestingly, 12.5% (n = 10) of the students scored marks between the highest mark (26-30). This indicates that the general performance of the final year (SHS3) students of MGSHS performed very well in the SOLO super-item test.

Table 5 presents item analysis of students and the percentages of the students reaching each SOLO level based on their responses to the five test items in each question.

Table 5: Overall Participants' Performance on each item in the SOLO levels

Item analysis of Uni-structural level for all participants

Test items	No. of students with correct answer (n)	%	No. of students with wrong answer (n)	%	Total (n)
1 (i)	77	96.25	3	3.75	80
2 (i)	69	86.25	11	13.75	80
3 (i)	46	57.50	34	42.50	80
4 (i)	44	55.00	36	45.00	80
5 (i)	23	28.75	57	71.25	80

Item analysis of Multi-structural level for all participants

Test items	No. of students with correct answer (n)	%	No. of students with wrong answer (n)	%	Total (n)
1 (ii)	70	87.50	10	12.50	80
2 (ii)	53	66.25	27	33.75	80
3 (ii)	61	76.25	19	23.75	80
4 (ii)	43	53.75	37	46.25	80
5 (ii)	39	48.75	41	51.25	80

Item analysis of Relational level for all participants

Test items	No. of students with correct answer (n)	%	No. of students with wrong answer (n)	%	Total (n)
1 (iii)	15	18.75	65	81.25	80
2 (iii) & (iv)	47	58.75	33	41.25	80
3 (iii)	54	67.50	26	32.50	80
4 (iii)	40	50.00	40	50.00	80
5 (iii) & (iv)	21	26.25	59	73.75	80

Item analysis of Extended abstract level for all participants

Test items	No. of students with correct answer (n)	%	No. of students with wrong answer (n)	%	Total (n)
1 (iv)	10	12.50	70	87.50	80
2 (iii)& (iv)	47	58.75	33	41.25	80
3 (iv)	49	61.25	31	38.75	80
4 (iv)	37	46.25	43	53.75	80
5 (iii)& (iv)	21	26.25	59	73.75	80

It can be observed from Table 5 that most students were able to score most of the items in the uni-structural and multi-structural levels. With the exception of item 5, more than half of the students scored the rest of the items 1, 2, 3 and 4 in both levels. In the relational level, students scored more than half of the scores in items 2 and 3 and exactly half of the students (n = 40) had item 4 correct, whilst 81.25% (n =65) and 73.75% (n = 59) could not answer the items 1 and 5. Students could not score well in the extended abstract level items, with only 12.5% (n = 10) of the students scoring all correct in item 1. The items (iii) and (iv) of questions 2 and 5 also tested up to the extended abstract level. Students could not handle item 5 at all in all the levels. The result in Table 5 is a clear indication that as students move from one thinking level to the other increasing manner of SOLO, then the performance of the number of students at the levels decreases.

Table 6 summarizes the students' performance into the various levels with percentage of students reaching each SOLO level of thinking in Circle Theorems. It was observed in Table 5 that most students scored majority of the items in uni-structural and multi-structural levels of SOLO. However, this observation does not reflect in the number of students who have attained the uni-structural and multi-structural levels. How to attain each SOLO level has been explained on Table 2.

Table 6: Distribution of Number of Participants reaching each SOLO level of thinking in Circle Theorems

SOLO Level	No. of students reaching level	%
Pre-structural	3	3.8
Uni-structural	8	10.0
Multi-structural	14	17.5
Relational	6	7.5
Extended	49	61.3
Total	**80**	**100.0**

The result in Table 6 shows that three students representing 3.8% remained pre-structural level of the SOLO levels of thinking in the four theorems explored in this study. However, 8 students representing 10.0% attained the Uni-structural level. Also, 14 students (17.5%) were able to attain the multi-structural thinking in Circle Theorems. Forty-nine (49) out of the 80 students representing 61.3% proceeded to extended abstract level of thinking with only 7.5% remaining at the relational level. Figure 5 is a graphical display of the proportion of participants who reached each level on thinking in circle theorem based on SOLO taxonomy.

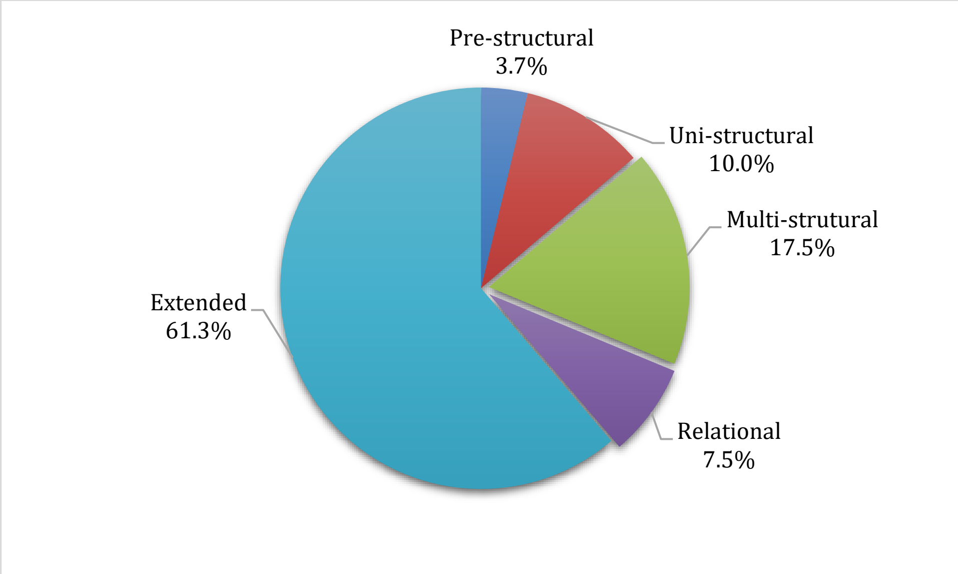


Figure 5: Proportions of participants reaching each SOLO level

4.2 Research question 2: How do SHS students think in solving problems in Circle Theorems?

In answering this research question, descriptive analysis and item analysis were both done to ascertain how students think in solving problems involving theorems on Circles. A descriptive analysis for each SOLO level was performed and the results presented in table 7.

4.2.1 Item Analysis

The theorems used in the study are numbered for the purpose of identifying them in this study only. Analysis of the number of participants who got the items on each theorem correct with their percentages is presented in Table 7.

Table 7: Overall Performance on each Theorem

Test items	Theorem	No. of students who had it correct (n)	%	No. of students who had it wrong (n)	%	Total (n)
1 & 2	1	49	61.25	31	38.75	80
3	2	53	66.25	27	33.75	80
4	3	41	51.25	39	48.75	80
5	4	26	32.50	54	67.50	80

<u>Theorem 1: angle subtended from a diameter is 90°</u>

Item 1 and item 2 both required students to identify the angle at the end of the diameter, and use it together with a few applications on the theorem: angle a chord or an arc subtends at the centre is twice the angle at the circumference (see Appendix D). Students were asked also to draw conclusions and provide reasons for their choices. The result of the analysis in Table 7 shows that out of 80 students, 31 representing 38.75% could not respond the items regarding this theorem. This is because majority (70%) could not explain the reasons in relation to their answers provided, which constitute the extended abstract level (see Table 5). Examples of students' work are exhibited in Figure 6 and Figure 7.

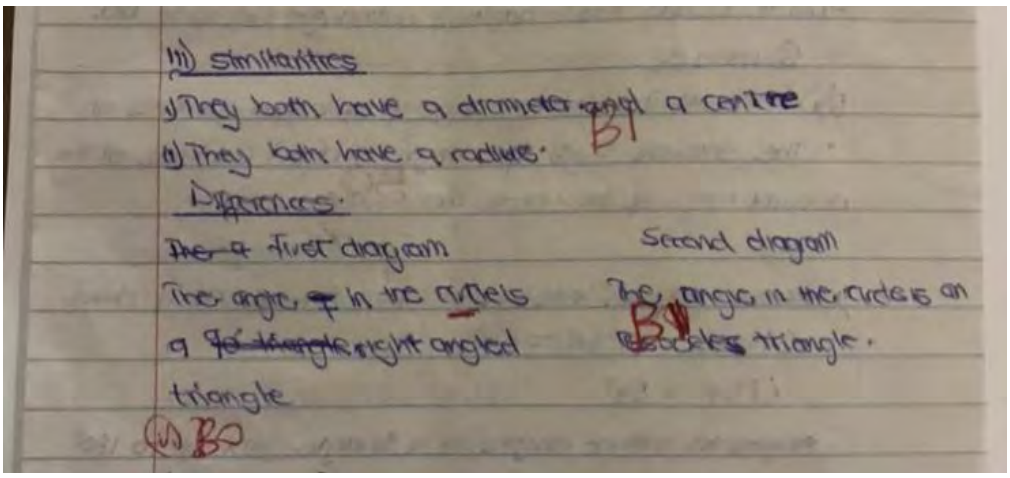


Figure 6: Example of Akosua's response on item 1(iv)

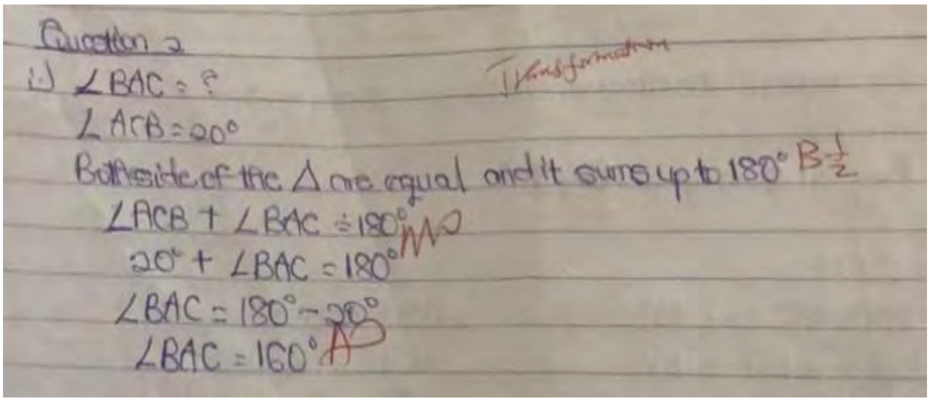


Figure 7: Example of Barikisu's response on item 2(i)

In Figure 6, Akosua's response shows that although she managed to get marks for the similarities and differences item, she could not provide reasons in relation to these answers. Also, the response provided by Barikisu, as shown in Figure 7 suggests that the student understood the question, however the answer was incorrect since the answer from the previous work was wrong for $\angle ABC = 90°$. Although students had an idea to use the sum of interior angle of a triangle, the substitution was wrong. This may be due to an error that can be corrected once the misconception between a diameter and chord is corrected.

It is therefore expected that the theorem stated above should clearly be explained to students to improve their problem-solving skills in Circle Theorem since this theorem serve as a prerequisite to the other theorems.

<u>Theorem 2: Angles subtended by the same arc or chord in the same segment are equal</u>

Most students (66.25%) representing 53 students were able to respond to the item regarding this theorem correctly as shown in Table 8, some were unable to solve the question at all. Again, 38.75% (n = 31) of the students couldn't draw conclusions to their choices (see Table 5). Examples of how some students executed their work is shown in Figure 8 and Figure 9.

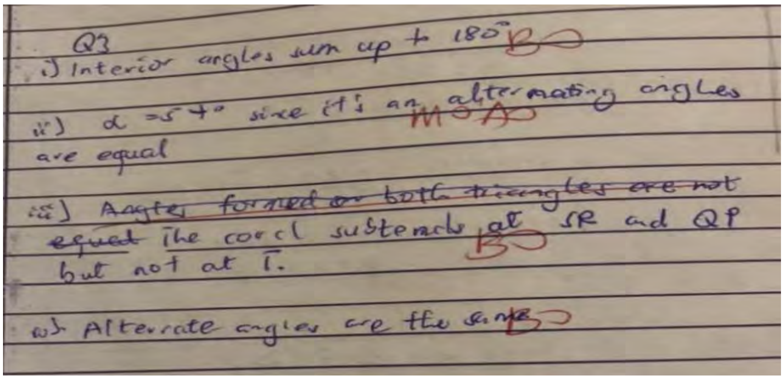


Figure 8: Example of Kuukua's response on item 3

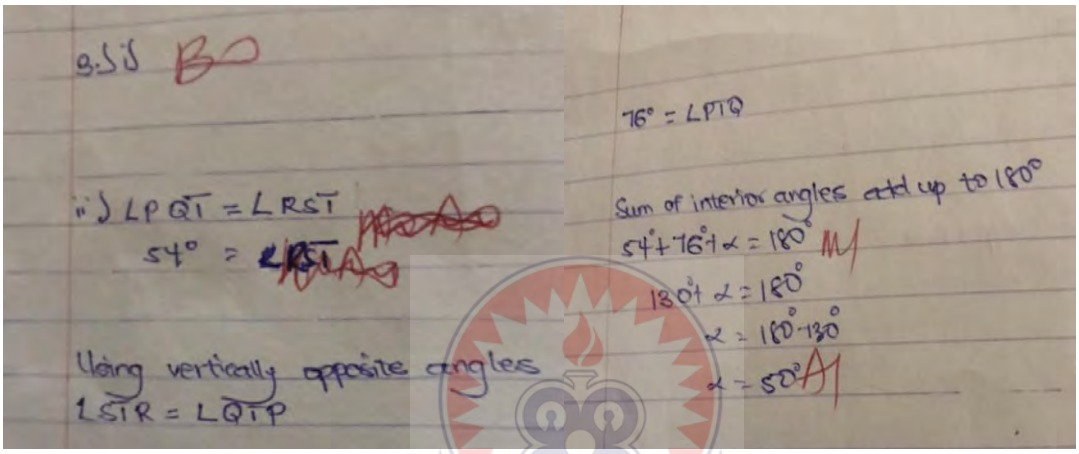


Figure 9: Example of Ama's response on item 3

The answer provided by Kuukua as shown in Figure 8 suggests that she did not know how to go about the question. The student got the theorem to be wrong as well as $\angle RST$ and $\angle PQR$. Nonetheless, she was able to identify that the sum of the interior angles of a triangle (Q3i) as shown in Figure 8, that would be used to get the missing angle. Although, Ama's response, shown in Figure 9 identifies that $\angle RST$ and $\angle PQR$ are equal and was able to solve for the missing angle, she could not arrange the answers correctly as they were asked. This shows that she knew the concept but had challenges in the procedure.

Most students understood the question well especially where they were to find the value of the missing angle (Q3ii. calculate the value of α) and stating the relationship between the angles at Q and at S (see Appendix C), but could not justify their reasons for their choices. Although, quite a number of students had it completely correct. This shows that only few of the students (n=27) do not understand the theorem and cannot apply it in solving problems.

Theorem 3: Angles in alternate segments are equal

The item which used this theorem required students to identify the equal angles in the diagram and explain why they are equal. Students were expected to use the identified angles to solve for the missing angle and give reasons for their option (See Appendix C). Almost all students responded to the item regarding this theorem, however, 48.75% of the students found it difficult to apply the theorem. Even though, some students provided reasons for the answers provided, some could not identify the angles that were equal. Example of a student work is shown in Figure 10.

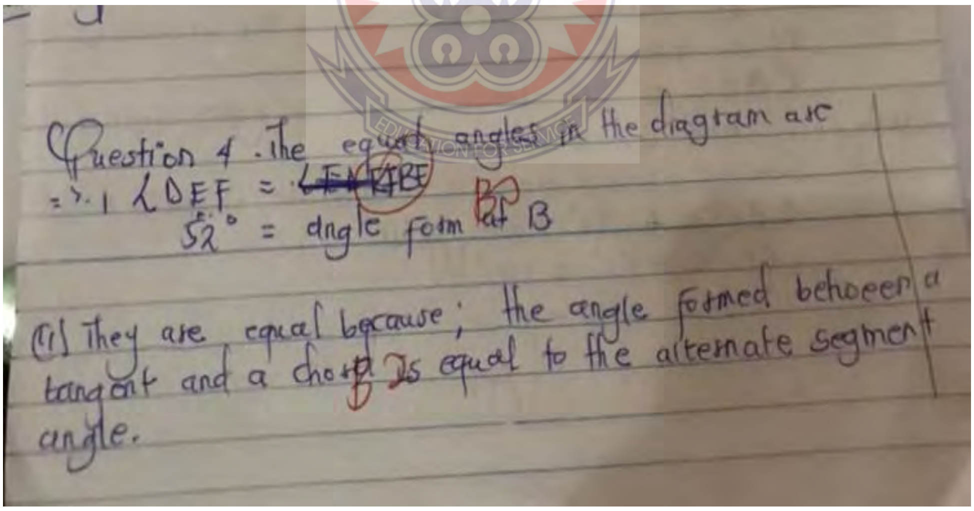


Figure 10: Example of Yaaba's response on item 4

In Figure 10, although, the student was able to remember and explain the equal angles and use the idea to solve for the value of the missing angle, she could not relate this idea in identifying the equal angles. This indicates that the conceptual understanding of the theorem: "angles formed between a chord and a tangent at the point of contact is equal to the angle formed in the alternate segment" was not understood well, since she could not relate it to the question.

Theorem 4: Angle formed between a tangent and radius (diameter) is 90°

This theorem was used in item 5 (see Appendix C), it demanded that student's state one theorem found in the diagram and use it to calculate for the value of a missing angle and give reason for their solutions. Students were required to recall their knowledge on properties of quadrilaterals in their previous studies to solve for the value of the missing angle. Table 8 shows that out of the 80 students, only 26 representing 32.5% were able to respond to the item regarding this theorem correctly. However, some were able to find alternate way to arrive at the answer, without stating the theorem involved. Example of how a student executed this item is shown in Figure 11.

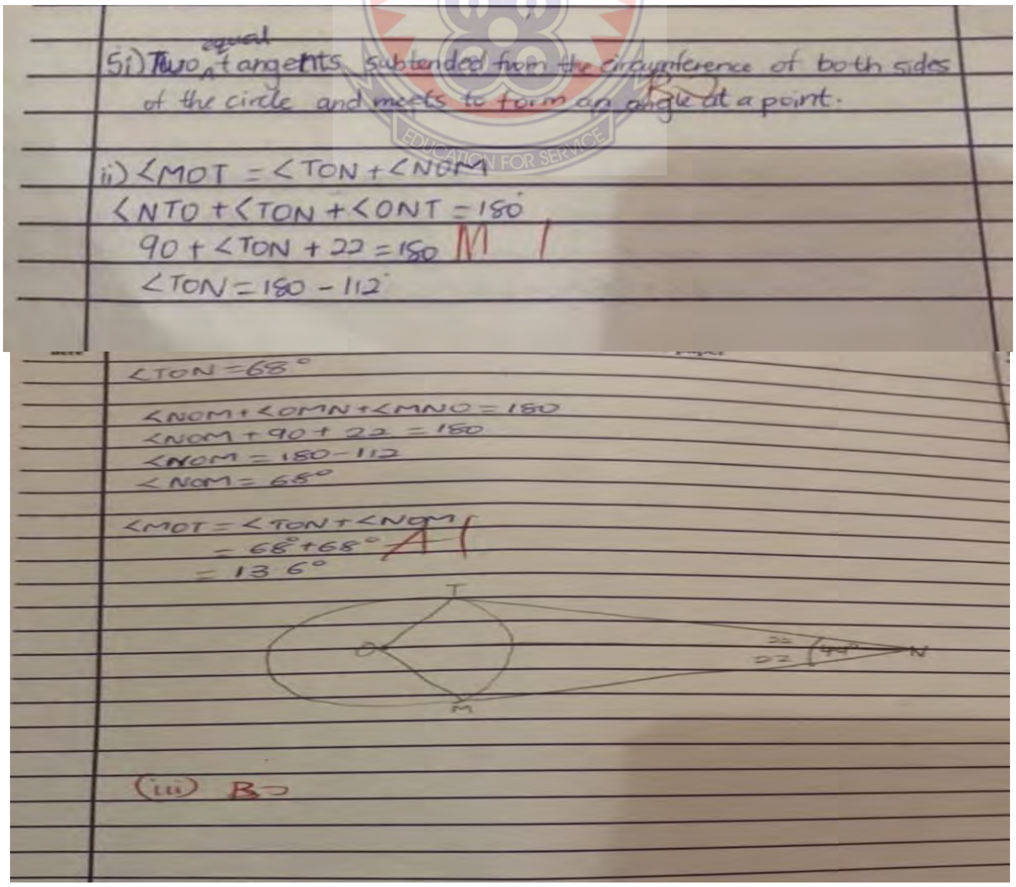


Figure 11: Example of Aisha's response to item 5

The student work in Figure 11 is an evident that she had the ability to solve for the value of the missing angle by applying the sum of interior angles of a triangle and then multiply the answer she got by two. She knew that the length of the external lines touching the circle from the same point are equal and, therefore divided the quadrilateral into two and solve for one side, and added the angle twice.

Generally, most of the students had poor understanding of the question and therefore did not attempt the question at all. Those who were able to answer it, only 23 representing 28.75% of them (see Table 5) were able to state the theorem to be used (see Appendix D). This indicates that students do not under the theorems involving tangents in Circle theorem well.

4.3 Research question 3: What errors and misconceptions do SHS students commit when solving problems on Circle Theorems?

In answering this research question, errors and misconceptions were defined according to literature and error analysis was done to identify the various errors committed by students in solving problems on Circle Theorems. These errors were analysed based on NEA framework. Unstructured interview was conducted to highlight some causes of misconceptions SHS students have on solving problems on Circle Theorems in this section.

4.3.1 Results from Students' Interview sessions

The interview (see Appendix E) held after answering the test item allowed students to give detailed responses in explaining their solving abilities within the four types of SOLO taxonomy in each question item. They added more to their initial responses when queried and prompted in an interview situation but in many instances, their responses were still at the same level rather than increasing the level of the responses.

The interview session discovered many misconceptions in solving problems on Circle Theorems held by the students. As quoted by Swan (2001) cited in Luneta (2015), "misconceptions are an integral part of learning a concept". As students develop a more robust understanding of the concept, the misconceptions will be substituted by more comprehensive and suitable conception.

The interview response from item 1 showed that even though, students knew a chord touches the endpoints of a circle, some were not certain about how to apply it to the question given. This brought about a misconception which is, some students not knowing that diameter is also a chord.

The interview session (see Appendix E) on item 2 revealed that the student had the concept relationship correctly described but failed to recognise that the diagrams in both question 1 and question 2 are the same, just an additional information has been given. Though she had angle ABC in question 1 correct, she could not link that idea to question 2. She did not really know what a radius is but considered lines AB and BC to be radius and therefore thought the triangle is isosceles.

The interview result from item 5 (see Appendix E) indicated that the student applied a well-known fact to angles, which are angles subtended by an arc or chord at the centre is twice that at the circumference of a circle, but had a misconception about the angle formed in this question. The result would have been true, if $\angle MNT$ is on the circumference of the circle. Students with this type of misconception believe that angle formed at the centre of a circle is always twice the angle formed whether the angle formed is on the circumference or not. As long as the centre angle is there, they must equate it to twice an angle formed. Out of 41 students who did not get this item correct, nearly 20 of them had same misconceptions as Akuba (see Appendix E).

However, it is important for teachers to help their students become aware of the misconceptions they have, so that these can be tackled and resolved.

4.3.2 Error Analysis of students' responses to the test items

Test items in this study were marked with attention being paid on the several errors made by learners using Newman's Error Analysis (NEA) model. These errors were analyzed based on learners' response in answering each test item. NEA is a framework used to identify and analyze learners' errors in solving mathematical problems (Newman, 1977 cited in Fitriani, Turmudi & Prabawanto, 2018). The errors are classified according to Newman as reading error, comprehension error, transformation error, process skills error and encoding error.

In this study, the errors and misconceptions made by students varied. They comprise of students poor understanding and identifying of the basic definitions of parts of circles; inability to recall previous knowledge on angle properties and remember the correct theorems to use, all these are classified under comprehension errors in NEA framework; inappropriate choosing of property (theorems) and relating it to solve the given problems (transformation error) and also inability to use the correct angles; inability to use right procedures or incorrect operations or wrongful calculations (process skills error) and lastly, inability of students to justify or draw conclusions in relations to their responses (encoding error).

This study focused on four types of NEA framework: comprehension, transformation, process skills and encoding. The analysis of the number of students' errors based on their responses are presented in Table 8.

Table 8: Analysis of students' errors based on NEA framework

	Students who committed errors	
Error	**Number**	**Percent**
Comprehension	43	53.75
Transformation	29	36.25
Process skills	3	3.75
Encoding	68	85

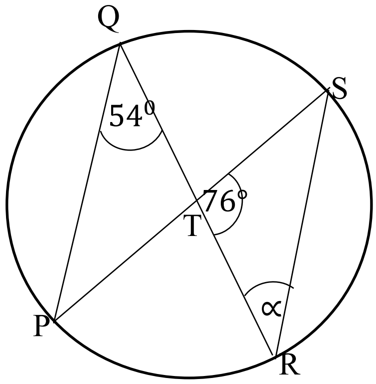
Using NEA for the analysis of the questions (items), students seem to be able to read and decode the questions correctly. However, the students had problems with comprehension, transformation, process skills and encoding errors. The thematic analysis of students' responses is presented as follows:

Theme 1: Comprehension error

The study identified comprehension errors in students poor understanding and identifying of the basic definitions of parts of circles; inability to recall previous knowledge on angle properties and remember the correct theorems to use. From Table 8, at the comprehension stage, 53.75% of the students failed to understand the needs of the questions. The students seemed to not have idea on how to solve the questions (example is shown in Figure 12), even though some manage to know that sum of interior angles of a triangle is 180°. This is what transpired in an interview with one of the students:

The item needed the students to define the theorem that will be used to find the angle $\angle RST$ and also state the relationship between the angles $\angle RST$ and $\angle PQR$ and use it to calculate for a missing angle. Most of the students had the angle QRS to be wrong. In the following extract, Kuukua tries to explain her solution:

Researcher: *How did you get* $\angle QRS = 54°$ *in the diagram?*



Kuukua: $\angle QRS = 54°$ *because it's an alternating angle and alternating angles are equal.*

Researcher: *How did you get to know the angles are alternating?*

Kuukua: *erm since the diagram is forming a 'Z' symbol that's why they are alternating.*

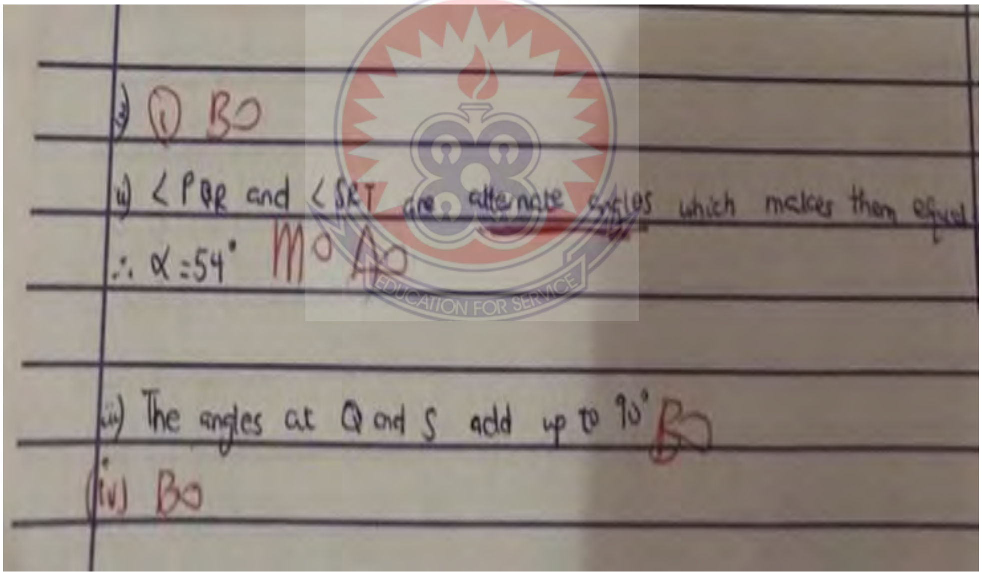


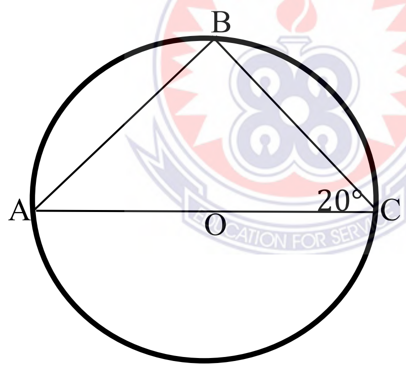
Figure 12: Example of a students' work on comprehension error

In this explanation, the student had been taught that an angle forming a 'Z' shape is an alternating angle but she did not really understand the concept well. Maybe, the teacher did not demonstrate it well for them to understand. The student's lack of understanding of alternating angles revealed that there has been a misconception and therefore needs to be corrected.

Theme 2: Transformation error

At the transformation stage, inappropriate choosing of property (theorems) and relating it to solve the given problems were identified in the study. Table 8 shows that 29 students representing 36.25% were unable to use the correct theorems to solve the questions. However, the students still found problems in determining the correct angles to use (example is shown in Figure 13). For instance, the questions in item 2 demanded the application of the knowledge of the theorem used in the previous question. Responses from the students indicated that most students could not identify that $\angle ABC = 90°$ in the diagram and therefore had the solution to be wrong. An interview with Ayorkor regarding item 2(i) is as follows:

Researcher: *How did you get your solution in item 2(i)?*



Ayorkor: *Triangle ABC is an isosceles triangle, which means* $\angle ACB$ *and* $\angle BCA$ *are equal base angles. This implies that* $\angle BCA = \angle ACB$. *Therefore, knowing that the interior angle of a triangle adds up to* 180. *I added* $\angle ACB$ to$\angle BAC$ *and equated them to* 180°. *since one angle of the base has been given to be* 20°, $180° - 20° = 160°$.

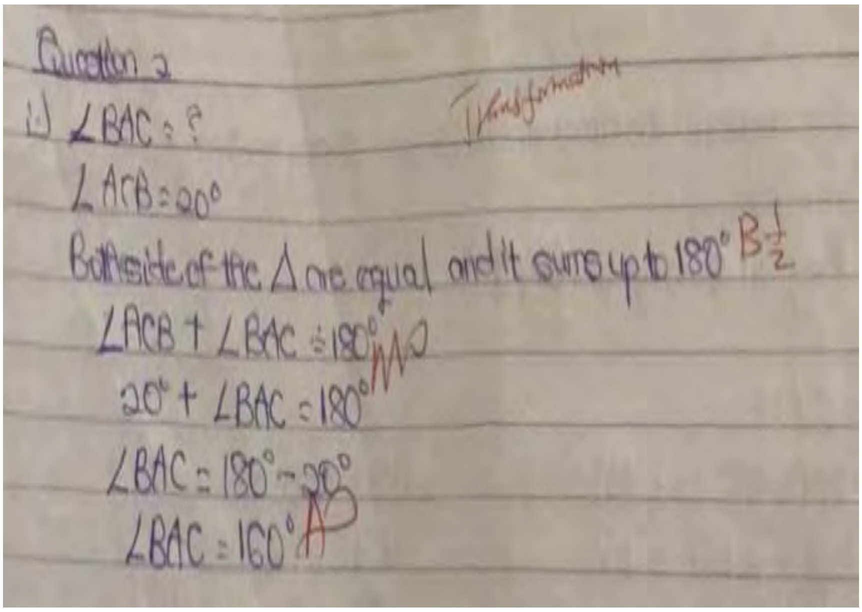


Figure 13: Example of students' work on transformation error

Researcher: *But a triangle has three sides, so why did you add* $\angle BAC$ *to* $\angle ACB$ *and subtracted from* 180°?

Ayorkor: *I didn't know what to do, but I remembered interior angles of a triangle add to* 180° *so I used it since the triangle is isosceles.*

Researcher: *why is the triangle isosceles?*

Ayorkor: *Because* $AB = BC$ *since they are both radii.*

From the interview session, the student had the concept relationship correctly described but failed to recognise that the diagrams in both question 1 and 2 are the same, just an additional information has been given. Though she had angle ABC in question 1 correct, she could not link that idea to question 2. She did not really know what a radius is but considered lines AB and BC to be radius and therefore thought the triangle is isosceles.

Theme 3: Process skill error

At the process skill stage, students' inability to use the correct angles; inability to use right procedures or incorrect operations or wrongful calculations were identified in the study. Table 8 indicates that 3.75% of the students were unable to solve the procedures correctly (example is shown in Figure 14, the student used wrong calculation).

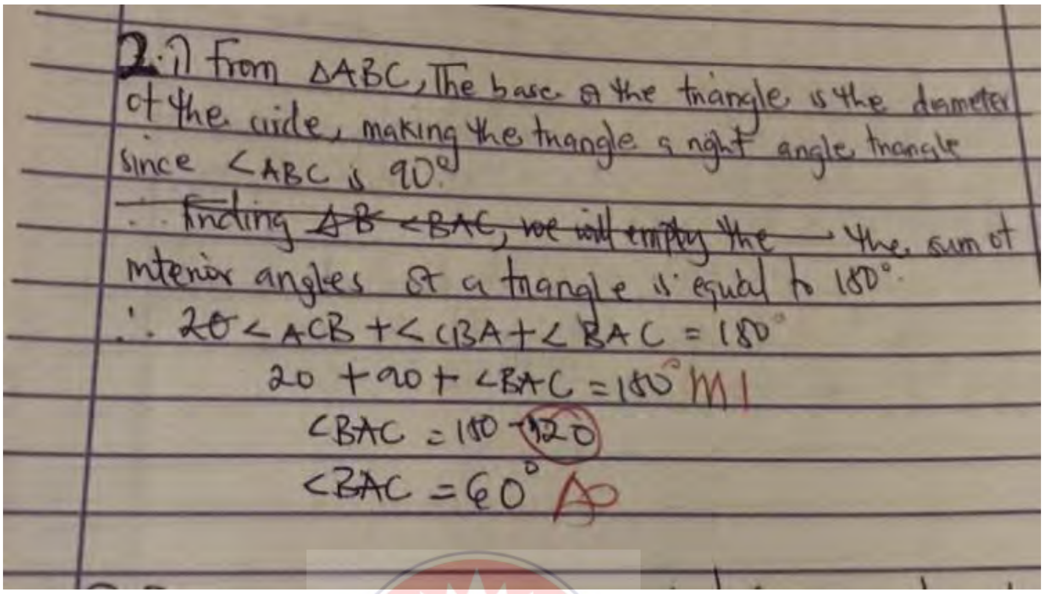


Figure 14: Example of students' work on process skills error

Theme 4: Encoding error

At the final stage, encoding stage, students exhibited their inability to justify or draw conclusions in relations to their responses. From Table 8, majority of the students representing 85% seem to be confused on drawing conclusions or giving reasons to their solutions (example is shown in Figure 15, misunderstanding the question).

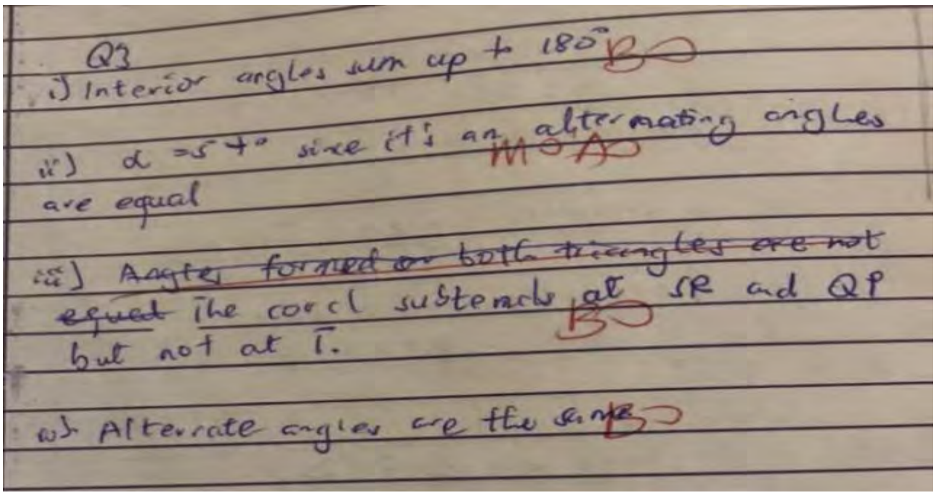


Figure 15: Example of a students' work on encoding

The responses students gave showed that they were not able to understand what they have solved although their answers were correct. This implies students cannot give reasons or justify their final answers correctly.

4.4 Research question 4: Do SHS students' demographic variables (age, programme of study and location) influence their level of thinking on solving problems on Circle Theorems?

In Research Question 4, the researcher sought to determine if participants' levels of thinking are influenced by their demographic variables (age, location and programme study). The following null hypothesis was formulated and tested for this question:

H_0: There is no significance difference between students' age, programme of study and location and their thinking level on Circle Theorems.

The mean and standard deviation for the location and thinking levels of students are presented in Table 9.

Table 9: Group Statistics

Level	Location	N	Mean	Std. Deviation
Uni-structural	Urban	54	4.7500	1.09351
	Rural	26	4.6923	1.12318
Multi-structural	Urban	54	5.5370	2.22942
	Rural	26	6.1731	1.95930
Relational	Urban	54	4.7963	2.24978
	Rural	26	5.1538	1.82082
Extended abstract	Urban	54	2.3889	1.25016
	Rural	26	2.5000	1.33417

Table 10: Independent Samples Test for difference in thinking level based on location of the students

Level		Levene's Test for Equality of Variances		t-test for Equality of Means		
		F	Sig.	T	df	Sig. (2-tailed)
Uni-structural	Equal variances assumed	.066	.798	.219	78	.827
Multi-structural	Equal variances assumed	1.149	.287	-1.241	78	.218
Relational	Equal variances assumed	1.560	.215	-.706	78	.482
Extended abstract	Equal variances assumed	.061	.805	-.364	78	.717

The Levene's test for equality of variances was used to test for locality (urban and rural) of the participants to determine whether the assumption of the equality of variance were violated. The results from Table 10 indicate that equality of variance was assumed for all groups in the students' thinking levels based on SOLO taxonomy. This means that there is no significant difference in the thinking levels of students who lived in the urban and rural areas.

Tables 11 and 12 show a one-way ANOVA conducted to compare the relationships between the participant's age and programme of study and their thinking levels. The observations from both analysis of variance indicated that results were independent. There were no outliers and no multicollinearity (all were examined and not violated). This can be explained that the age group or the programme of study of each participant and their thinking levels was not significant. There is no relationship between these variables.

Table 11: One-way ANOVA statistics for difference in thinking level based on age of students

Level		Df	Mean Square	F	Sig.
Uni-structural	Between Groups	4	.597	.484	.748
	Within Groups	75	1.234		
	Total	79			
Multi-structural	Between Groups	4	3.358	.713	.585
	Within Groups	75	4.708		
	Total	79			
Relational	Between Groups	4	10.117	2.425	.055
	Within Groups	75	4.172		
	Total	79			
Extended abstract	Between Groups	4	2.283	1.446	.227
	Within Groups	75	1.579		
	Total	79			

Table 12: One-way ANOVA statistics for difference in thinking level based on student programme of study

Level		Df	Mean Square	F	Sig.
Uni-structural	Between Groups	2	.815	.672	.514
	Within Groups	77	1.212		
	Total	79			
Multi-structural	Between Groups	2	32.447	8.284	.001
	Within Groups	77	3.917		
	Total	79			
Relational	Between Groups	2	6.187	1.397	.254
	Within Groups	77	4.429		
	Total	79			
Extended abstract	Between Groups	2	2.183	1.365	.262
	Within Groups	77	1.600		
	Total	79			

4.5 Discussions of Findings

This section presents the discussions on the findings and the implications of these findings. The findings are discussed according to the research questions as listed in section.

4.5.1 Research question 1

The first research question was: What thinking level with respect to SOLO taxonomy, do SHS students attain in solving problems on Circle Theorems? This question was answered by using the scores of the students in the cognitive test on Circle Theorems.

To answer the question, descriptive statistics were computed and students' results analysed. The result in Table 6 revealed that three students representing 3.8% failed to attain any of the SOLO levels of thinking in the four theorems explored in this study. There were 77 of the students representing 96.3% who attained Uni-structural level. However, 8 students representing 10.0% could not proceed to the next level of multi-structural thinking. This revealed that students were more successful in answering questions in the Uni-structural level. Also, out of the 69 students (86.3%) who attained multi-structural thinking in solving problems on Circle Theorems, 55 representing 68.9% proceeded to Relational thinking level with only 14 students representing 17.5% remaining at that level. This also indicated that students performed better in the multi-structural level and gradually proceeded to the Relational level of thinking. Again, out of the 55 students (68.8%) who attained Relational thinking, 49 of them representing 61.3% proceeded to extended abstract level of thinking with only 7.5% remaining at the Relational level. This showed that most students in the Relational thinking level were able to attain to the level of

Extended abstract.

The findings revealed that most students were operating at the higher cognitive levels (Extended abstract levels) of the SOLO taxonomy guide (see Figure 5). This suggests that students have moved beyond the surface (content) knowledge, they are operating in the qualitative phase of the SOLO taxonomy as stated in the literature. This finding is in non-concurrence with the findings of Boulton-Lewis (1992) and Boulton-Lewis and Dart (1995) cited in Lucander, Bondemark, Brown & Knutsson (2010), which found out that the results of their studies in terms of improving students' structural organization of knowledge were disappointing because majority of the students remained in the multi-structural levels of the SOLO taxonomy.

The research showed that most students were able to attain the highest levels of the SOLO taxonomy. This finding implies that most of the students' level of thinking are higher as required by the mathematics curriculum, which states that students should be able to develop precise, logical and abstract thinking (MOE, 2010). These indications show that student have reached the stage of reasoning deductively in solving problems on Circle Theorems.

4.5.2 Research question 2

The second research question was: How do SHS students think in solving problems in Circle Theorems? To answer this research question, item analysis was done to ascertain how students think when solving problems in Circle Theorems.

The analysis on the test items is discussed according to the student's responses to the theorems in the study.

Theorem 1: Angle subtended from the diameter is 90°

This theorem was used for both item 1 and item 2. The analysis on item 1 revealed that of the 80 participants who answered the item, 70 (87.5%) of them could not get the answers to the question on the Extended abstract level; 77 (96.3%) got the question correctly answered in the Uni-structural level and 8 (10%) had attempted the question but got none correct in the multi-structural level. In item 2, Relational and Extended abstract levels question was combined. Out of the 80 participants, 33 (41.3%) of them did not attempt the question on the Relational and Extended abstract levels whilst 69 (86.3%) got all the answers correct in the Uni-structural level. About 3 or 4 students had the answers partially correct in all the levels and 27 (33.8%) had none of the answers correct in the Muti-structural level.

In general, most of the students 49 (61.3%) solved the items involving this theorem correctly with few exceptional cases; where some students solved part of the items correct and the rest of the procedures wrongly. The students showed their abilities on the calculation part of solving the problems than the drawing of conclusions or reasons given to their choices. These students were unable to inter-relate the theorems or properties to be used to their calculations. This finding concurs with the findings of Lim & Idris (2006), which stated that the low ability students do not make little or no connections to the contextual aspect of the data though they used both the visual and qualitative aspect of the data consistently.

Theorem 2: Angles subtended by the same arc or chord in the same segment are equal

The performance of students in this aspect performed better than the other theorems used in this study (see Table 7). Students who answered all the questions correct in the Relational level numbered 67.5% and those who answered the question wrongly in the Extended abstract level numbered 38.8%. In all 53 students representing 66.3%, showing more than half of the students were able to answer the question regarding this theorem.

Theorem 3: Angles in alternate segments are equal

The analysis in this item revealed that out of 43 students representing 53.8%, 37.5% of them who took part in the study did not attempt the question in the Extended abstract level at all whilst 55% of them answered the question in the Uni-structural level correctly. 40% of the students attempted the question but had their solutions to be wrong in the Relational level and 21.3% of them got partially correct answer.

Theorem 4: Angles formed between a tangent and radius (diameter) is 90°

In this aspect, the Relational and Extended abstract levels were combined in one question. It was revealed that 73.8% of the participants could not answer the question in the Relational and Extended abstract levels whilst 28.8% got the answer correct in the Uni-structural level. In all, 54 of the students representing 67.5%, which is more than half of the students could not answer or some cases attempt the item regarding this theorem. The performance of students in this section was not appreciable.

The research finding was clear that most students could not cope with questions involving reasons or drawing conclusions or even stating the circle theorem. These indicated the variation between student's thinking and the difficulty in reasoning deductively when solving problems on Circle Theorems. These results also have made known that students' knowledge and understanding of Circle Theorems was very limited. The main difficulty of students that resulted in their poor performance in the test was their inability to give reasons for each step of their solutions in a separated question in the test. It was also clear during the analysis of the students' solutions that, either they misrelated the theorems to their equations or miss out the key terms, or did not even indicate a theorem at all. This resulted in most students scoring low marks due to their understanding of the theorems based on memorization without understanding the rationale behind the algorithms given by the teacher. This finding is in line with Atebe and Schäfer (2011), who found that most African high schools' teaching remains conventional and tends to make students unable to solve variety of problem involving geometry.

4.5.3 Research question 3

The third research question was: What are students' errors and misconceptions in Circle Theorems? In answering this research question, error analysis and students' interviews were analysed.

The interview was conducted to provide more explanations on how students were thinking when solving each item in the super-item test according to SOLO taxonomy.

Analysis from the interview session with eight of the participants revealed that students gave detailed responses in explaining their solving abilities within the four types of SOLO taxonomy in each question item. They added more to their initial responses when queried and prompted in an interview situation but in many instances, their responses were still at the same level rather than increasing the level of the responses. The interview session also discovered many causes of the misconceptions in solving problems on Circle Theorems held by the students. As quoted by Swan (2001) cited in Luneta (2015), "misconceptions are an integral part of learning a concept". As students develop a more robust understanding of the concept, the misconceptions will be substituted by more comprehensive and suitable conception. However, it is important for teachers to help their students become aware of the misconceptions they have, so that these can be tackled and resolved. It was also revealed that students' inability to find solution was due to lack of basic knowledge for Circle theorem topic.

The errors made were also analyzed based on Newman's Error Analysis. Table 8 revealed that students had poor understanding and identifying of the basic definitions of parts of circles; inability to recall previous knowledge on angle properties and remember the correct theorems to use, inappropriate choosing of property (theorems) and relating it to solve the given problems and also inability to use the correct angles; inability to use right procedures or incorrect operations or wrongful calculations and lastly, inability of students to justify or draw conclusions in relations to their responses to the test items. This made the students commit errors in the second, third, fourth and fifth stages of the Newman's model, which is comprehension, transformation, process skills and encoding errors, with the most occurring one being encoding errors.

Some causes of errors and misconceptions were identified in the students' responses to item 1, which showed that students committed comprehension and encoding errors, with encoding error taking the greater number. Students' problems were a clear indication of their weakness in identifying some basic concept of Circles. Also, they tended to give wrong reasons to their working procedures or in some cases failed to give any at all.

The analysis on test item 2 revealed that students made errors in comprehension, transformation, process skills and encoding. Most of the students had errors in the comprehension error. These errors on comprehension were identified to be a misconception derived from the students' lack of understanding of the basic knowledge on Circles. These are misconceptions that need to be rectified. However, the process skills error was mistakes students made in their addition process and this kind of error can be solved easily.

In item 3 the errors identified were comprehension and encoding errors. Errors were made because students could not state the theorem to be used correctly. Some used the properties of angles “alternating angles”, which led to a misconception. These misconceptions need to be correct by mathematics teachers.

The performance of students in answering the item 4 was not without errors and misconceptions. It was detected that most students could not differentiate the theorems that state angles formed by a diameter and a tangent is 90° from angles in alternating segments are equal. This was a misconception that teachers are advised to assist students in some terminologies involving Circles.

Three types of errors were identified in item 5 namely, comprehension, transformation and encoding errors, with majority in the encoding error. The misconception found here was student’s understanding of the theorem “angle subtended by an arc at the circumference is twice the angle formed at the center”. The student’s lack of understanding of the concept of Circle theorems led to this misconceptions and errors.

The findings in this study revealed that most errors made by students were comprehension and encoding errors, with encoding error occurring most. Despite majority of the students committing errors in the encoding stage, they seem to demonstrate higher order of thinking in regards to the SOLO taxonomy. This is an indication that students have problems understanding the concept of circle theorems especially in the application of the right theorem, and writing the correct procedures involved in reaching their answers. This finding is in accordance with the study of Abduls, Nur Liyana and Marlina (2015), which they found out that, students commit errors and cannot answer questions because they do not understand the concept and thus, cannot interpret the needs of the question.

4.5.4: Research question 4

The fourth research question was: do SHS students' demographic variables (age, location and programme of study) influence their thinking levels? To answer this question, a null hypothesis was formulated and tested using independent t-test for the location, and one-way ANOVA for the age and programme of study.

The analysis from the hypothesis presents the relationship between level of thinking, and participants demographic variables (age, location and programme of study), which also answers research question 4. It was revealed that there is no statistically significant difference between participants thinking level and location and programme of study as well as age. This means that student's location, whether one is from urban or rural does not have any impact on their thinking levels, they all think at the same level. Also, the student's programme of study and age does not also affect their thinking in solving problems on Circle Theorems.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.0 Overview

This chapter presents a review and summary of the study, gives conclusions and makes recommendations mainly for educational purposes. In addition, suggestions for future research studies are presented.

5.1 Summary of the Study

The study aimed at assessing SHS students' thinking level using SOLO taxonomy guide in solving problems in the concept of Circle Theorems before writing WASSCE. Three research objectives and questions were analyzed. Data was collected using cognitive test from 80 SHS 3 students in MGSHS. The methodology used was the mixed method and the design used the sequential explanatory design.

The findings of the study revealed that:

- 49 of the participants representing 61.3% proceeded to extended abstract level of thinking with only 7.5% remaining at the relational level.
- Student's thought on solving problems on Circle Theorems is below average, which makes it difficult for them to reason deductively.
- Students performed better in the third theorem according to this study, which states, "angles subtended by the same arc or chord in the same segment are equal" than the fourth theorem, stated as "angle formed between a tangent and radius is 90°.
- Students made errors and misconception when solving problems on Circle Theorems.

- Four types of NEA were identified, namely, Comprehension, Transformation, Process skills and Encoding errors.
- Student's age group, programme of study and their locality do not have any effect on their level of thinking when solving problems on Circle Theorems.

5.2 Conclusion

A wide range of research argues that geometric thinking remains an essential part of mathematics responsible for the competencies of students in solving problems. In this study, it is quite evident that majority of the third-year students in MGSHS were successful in questions pertaining higher cognitive demand, i.e., Relational and Extended abstract levels of SOLO taxonomy. The findings of this study suggest that student's level of thinking in solving problems in Circle Theorems have reached the highest cognitive domain. This was indicated in their variation of performance in answering questions in the test items. Errors made were detected from the misconceptions students had arising from the poor understanding of some of the questions and weakness in the conceptual knowledge of geometry. It is therefore important to use SOLO taxonomy as an assessment tool to provide teachers with useful background on students' initial solving ability to enable them monitor the general growth of their student's solving ability.

5.3 Recommendations

The findings of this study suggest that SHS mathematics teachers need to pay attention to alternative ways of assessing students and the level of thinking each student is operating. It is therefore recommended that mathematics teachers need to use SOLO taxonomy as an assessment tool to provide teachers with useful background on students' initial solving ability to enable them monitor the general growth of their student's solving ability.

Mathematics Teachers should also take keen interest in the level of thinking of their students in order to identify and plan an appropriate intervention to address the student's difficulties. The use of SOLO taxonomy can be helpful on this path to remind teachers to focus on the cognitive abilities of their students.

Also, mathematics teachers should be interested in identifying errors and misconceptions student commit, and either minimize or eradicate them.

It is also recommended that content workshops and professional development should be organised for teachers to improve their teaching methodologies and way of assessing students based on SOLO taxonomy. This will enable teachers to know the level at which each student is operating.

Textbook authors are also recommended to pay attention to activities that involve higher order of thinking, for example analyzing, generalizing, evaluating and drawing conclusions; as well as varied questions that involve terminologies and notations in the subject. It would be helpful if teachers and textbook authors present questions in a more practical form than the abstract presentations to students. This would help the students in appreciating and enjoying the topic.

5.4 Areas for Future Research

This study did not reveal the cognitive barriers encountered when solving the problems on Circle Theorems according to SOLO taxonomy. Future research studies may be needed to address these issues so that the framework will be effective for supporting instructional programs that build on students' prior knowledge, nurture their ability in problem-solving and monitor their understanding. Future studies also need to be done to investigate whether the framework is appropriate for studies in other grade levels and different mathematics topic to determine the extent to which it can actually be used to inform instructional and assessment programs in SHS geometry (Circle Theorems). Future studies may be needed to ensure the effectiveness of the framework on a larger population and gender balance at different schools, since this study was limited to smaller scope and gender bias.

5.5 Limitations of the Study

Limitations in research are the shortcomings, conditions or influences that the researcher cannot control and place restrictions on the methodology and conclusions. Based on this, the sampling of this research was based on only three classes (General Science, General Arts and Business) due to the financial aspect of the study; the researcher could not involve more classes. Hence the scope and scale of this study is limited; conducting large-scale experimentation using larger samples than this one should test the validity of this study's results further.

Regarding the topic Circle theorem, it was supposed to be taught at SHS 2 according to the mathematics syllabus (MOE, 2012) but due to Covid 19 the topic was taught in SHS 3. This made the researcher to use only one school since most schools had not taught the topic at the time of the study. Hence to enhance the findings of the research, it would be prudent to use more schools including co-education and single-sex (boys) schools to validate the findings of this study.

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APPENDICES

Appendix A: Introductory Letter

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DEPARTMENT OF MATHEMATICS EDUCATION
P. O. Box 25, Winneba, Ghana. Tel: 233- 03323-20989, E-mail: maths @uew.edu.gh

9th June, 2021.

MFANTSIMAN GIRLS' S. H.S,
P.O.BOX SP 14, SALTPOND.
....
....
....

Dear Sir/Madam

INTRODUCTORY LETTER

Ms Victoria Felicia Aidoo-Bervell (200003166) is a graduate student who is pursuing an MPhiL. in Mathematics Education, at the Department of Mathematics Education, University of Education, Winneba. As part of her studies, she has been granted approval by the department's Graduate Board to undertake a research on **Using Structure of the Observed Learning Outcomes (SOLO) Taxonomy in assessing senior high school students' thinking levels in solving problems on circle theorems.**

We wish to introduce Ms Aidoo-Bervell to you for any assistance she may require to enable her gather data for her research.

We count on your consent and assistance.

Yours sincerely,
DEPARTMENT OF MATHEMATICS EDUCATION
UNIVERSITY OF EDUCATION
WINNEBA
Prof. J.M. Nabie
HoD
Department of Mathematics Education
University of Education, Winneba

APPENDIX B: Item Specification Grid

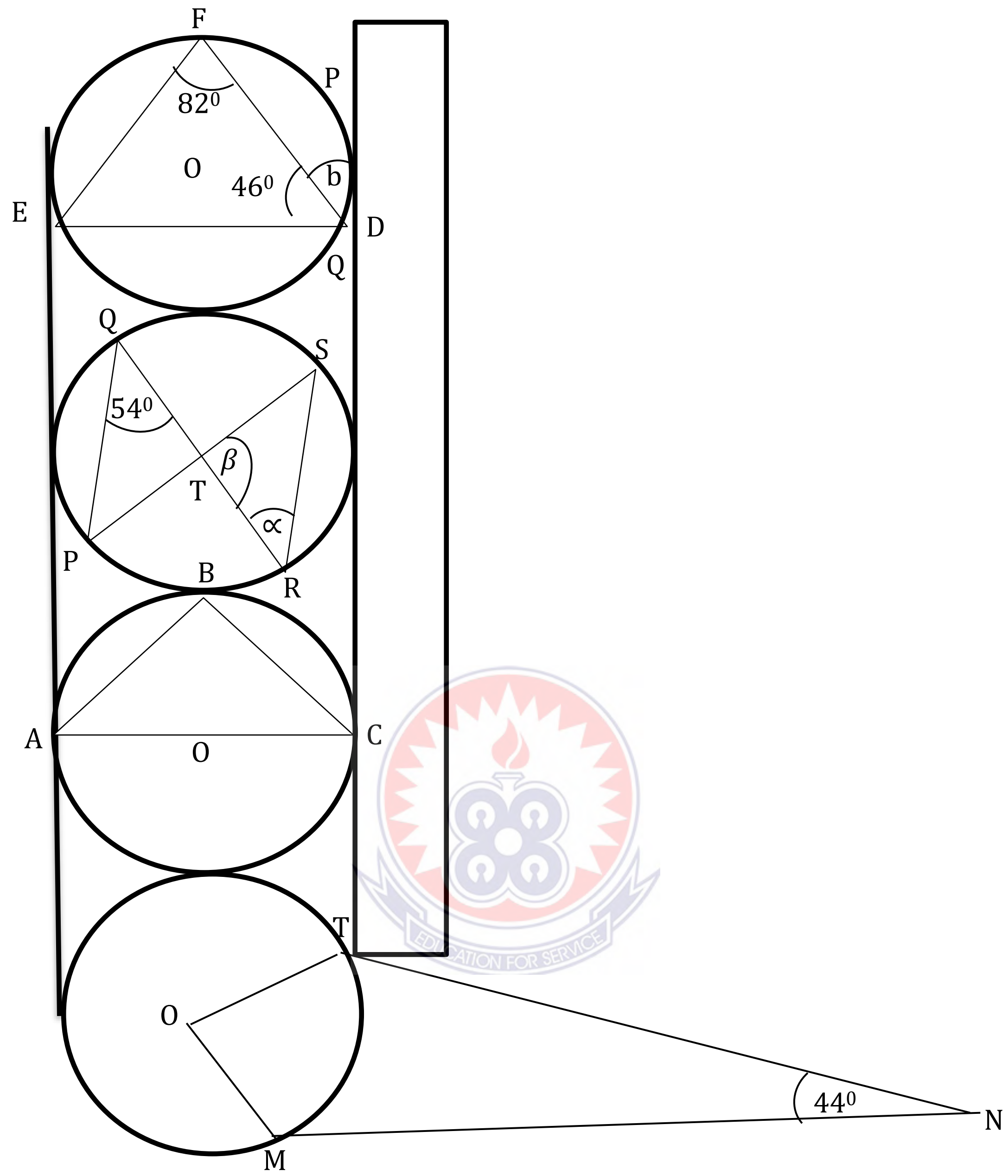
SOLO taxonomy	Key action verbs	Items	Content area tested	Scoring scheme
Uni-structural	Name, find, identify, define	1. i) What is the name of the chord that divides the first floor into two equal parts? 2. i) Find ∠BAC. 3. i) Define the theorem that will be used to find the angle at S 4. i) Identify the equal angles in the diagram. 5. i) Name one theorem that can be found in the diagram on the ground floor	1. Aspect of parts of a circle 2. Knowledge acquisition 3. Aspect of Circle theorems 4. Aspect of angle properties 5. Aspect of Circle theorems	1. 1 mark 2. 2 marks 3. 1 mark 4. 1 mark 5. 1 mark
Multi-structural	Describe, state, solve, calculate	1. ii) Using the name given in (i), describe the measure of angle formed at B in the first floor. 2. ii) State the theorem used to solve for the angle. 3. ii) Calculate the value of α. 4. Solve for the value of b. 5. Calculate the value of ∠MOT.	1. Conceptual understanding of Circle theorems 2. Aspect of Circle theorems 3. Aspect of triangle property 4. Procedural/solution process 5. Knowledge acquisition	1. 1 mark 2. 1 mark 3. 2 marks 4. 2 marks 5. 2 marks
Relational	Explain, compare and contrast, relate	1. iii) Compare the diagram of the first floor with the diagram drawn below with centre O by stating two differences and similarities. 2. iii) Give reason for your solution 3. iii) What is the relationship between the angles at Q and at S? 4. Explain why they are equal. 5. Give reason for your solution	1. Concept formation 2. Reasoning in Circle theorems 3. Knowledge acquisition 4. Knowledge acquisition 5. Reasoning in Circle theorems	1. 4 marks 2. 1 mark 3. 1 mark 4. 2 marks 5. 2 marks
Extended Abstract	Give reason, conclude	1. iv) What conclusion can be drawn from the diagrams above? 2. iv) Give reason for your solution. 3. iv) Give reason for your solution.	1. Inductive reasoning 2. Inductive reasoning in Circle theorems 3. Inductive reasoning	1. 2 marks 2. 2 marks 3. 2 marks

Appendix C: The Super-item test

Age: **Locality:** **Class:**

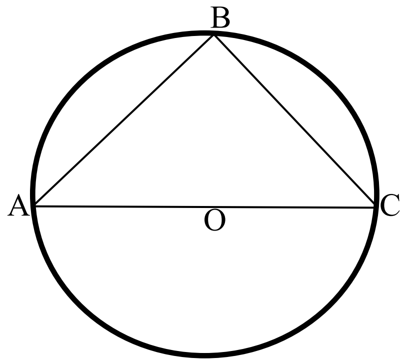
Answer All Questions

An architect wants to design a circular tower with three floors for an institution. From the entrance of the tower, there are two pathways, which are tangential to the ground floor of the circular tower. On the first floor she wants to partition the floor into two equal parts from the centre to the circumference of the floor. On the second floor, she will partition the floor with two intersecting inscribed angles. On the last floor, she wants to have major and minor segments with an inscribed triangle whose vertex will be protruded to develop a monument tangential to the wall.



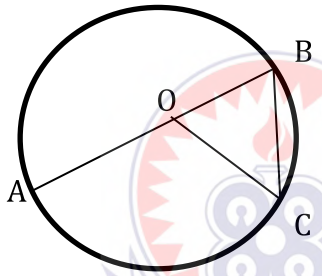
Use your understanding of the Circle Theorems in answering the following questions:

1. i) What is the name of the chord that divides the figure below into two equal parts?



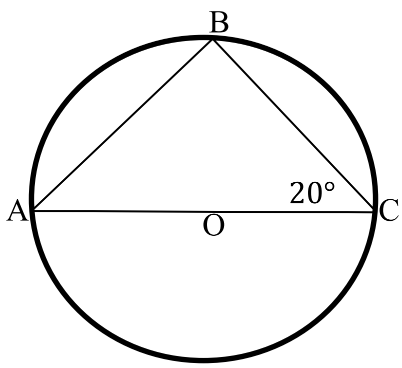
ii) Using the name given in (i), describe the measure of angle formed at B in the diagram?

iii) Compare the diagram above with the diagram drawn below with centre O by stating their differences and similarities;



iv) What conclusion can be drawn from the two diagrams above?

2.



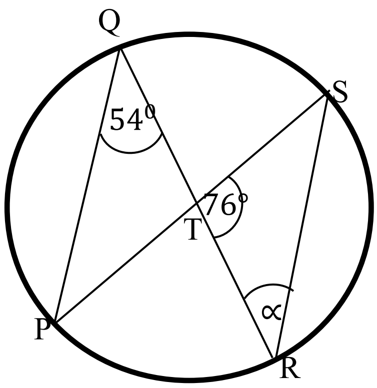
From the figure above, given $\angle ACB = 20°$.

i) Find $\angle BAC$.

ii) State the theorem used to solve for the angle.

iii) Give reason for your solution.

3. The figure shows a circular floor which has points P, Q, R and S on the circumference of the circle.



If it is given that $\angle PQR = 54^0$ and $\angle STR = 76°$, use the information to answer the following:

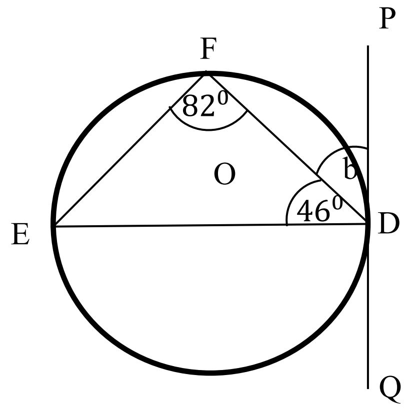
i) Define the theorem that will be used to find the angle at S.

ii) Calculate the value of α.

iii) What is the relationship between the angles at Q and at S?

iv) Give reasons to your solution in ii).

4. In the figure, DEF forms a triangle with PDQ as a tangent to the circle at D. If O is the centre of the circle, with $\angle EDF = 46°$ and $\angle DFE = 82^0$. Use the information to answer the following:



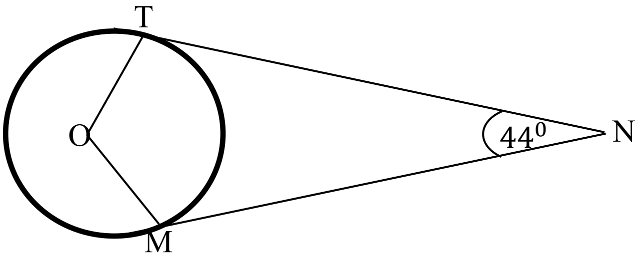
i) Identify the equal angles in the diagram.

ii) Explain why they are equal.

iii) Solve for the value of angle b.

iv) Give reasons for your answer in (iii).

5. The figure forms a circle with centre O, where M and T are points on the circle. If MN and TN are tangents to the circle with $\angle MNT = 44°$.



i) Name one theorem that can be found.

ii) Calculate the value of $\angle MOT$.

iii) Give reasons for your answer.

Appendix D: Solutions to the Super-item test

Question 1

I. Diameter …. **B1**

II. 90°… **BI**

III. Difference	Similarities
1. Triangle OBC is isosceles and triangle ABC is right angle. **B1** (**B**$\frac{1}{2}$ for any one)	1. AC is the diameter in both diagrams. **B1** (**B0** for wrong answer)
2. $\angle ABC = 90°$ in the diagram 1 but $\angle ABC = \frac{1}{2}\angle AOC$ in diagram 2. **B1**(**B**$\frac{1}{2}$ for any one)	2. $\angle AOC = 2\angle ABC$ in both diagrams. **B1** (**B0** for wrong answer)

IV. The theorem angle subtended by arc at the centre is twice the angle formed at the circumference can be used in both cases. **B2** (**B0** for wrong theorem).

Question 2

I. $\angle BAC + \angle ACB + \angle ABC = 180°$ but $\angle ABC = 90°$

$\angle BAC + 20° + 90° = 180°$ $\boldsymbol{M1}$

$\angle BAC + 110° = 180°$

$\angle BAC = 180° - 110° = 70°$

$\therefore \angle BAC = 70°$ $\boldsymbol{A1}$

II. Angle formed in a semi-circle is 90° **B1** (**B0** for wrong theorem).

III. Sum of interior angles of a triangle is 180° $\boldsymbol{B1}$ (**B0** for wrong reason).

i.e., $\angle BAC + \angle ACB + \angle ABC = 180°$ but $\angle ABC = 90°$

Question 3

I. Angles subtended by the same arc or chord in the same segment are equal. **B1**(**B**$\frac{1}{2}$ for not stating same segment).

II. $\angle RST + \angle STR + \angle TRS = 180°$

$\propto +76° + 54° = 180°$ $\boldsymbol{M1}$($\boldsymbol{M0}$ for wrong substitution)

$\propto +130° = 180°$

$\propto = 180° - 130° = 50°$

$\therefore \propto = 50°$ **A1** (**A0** for wrong answer).

III. They are equal. **B1** (**B0** for wrong answer)

IV. Sum of interior angles of a triangle is 180° and also

$\angle TSR = \angle PQR = 54°.\ \boldsymbol{B2}$ ($B1$ for stating only one).

Question 4

I. $\angle PDF\ and\ \angle DEF$ **B1** (**B0** for wrong answer).

II. Angle formed between a chord and a tangent at the point of contact is equal to the angle at the opposite (alternate) segment. **B2** (**B1** for not stating alternate segment).

III. $\angle DEF + \angle EFD + \angle FDE = 180°$

$b + 82° + 46° = 180°$ $\boldsymbol{M1}$ (M0 for wrong substitution)

$\Longrightarrow b + 128° = 180°$

$b = 180° - 128° = 52°$

$\therefore b = 52°\ \boldsymbol{A1}\ (\boldsymbol{A0}\ for\ wrong\ answer)$

IV. Sum of interior angles of a triangle is 180° and $\angle DEF = b$ since $\angle PDF = \angle DEF$ (angles in alternate segment are equal). **B2** (**B1**for any one)

Question 5

I. Angle formed by a tangent and a radius is 90°. **B1** (**B0** for wrong answer)

II. $\angle MOT + \angle OTN + \angle MNT + \angle NMO = 360°$

$\angle MOT + 90° + 44° + 90° = 360°$ $\boldsymbol{M1}$ ($\boldsymbol{M0}$ for wrong substitution)

$\Longrightarrow \angle MOT + 224° = 360°$

$\angle MOT = 360° - 224° = 136°$

$\therefore \angle MOT = 136°$**A1** (**A0** for wrong answer)

III. Sum of interior angles of a quadrilateral is 360° since MOTN is a quadrilateral. **B2 (B1** for any one).

APPENDIX E: The Interview Transcript

An interview with the students regarding the test items resulted in the following observations:

Responses from item 1 revealed that some students' lack the knowledge of what a chord is. This is a dialogue that depicted how one student responded to item 1(i):

Researcher: *What is a chord?*

Esinam: *A chord is like a diameter. It touches the two endpoints of a circle.*

Researcher: *From your work, you stated that "a chord from the diameter subtend and meet at the circumference". Can you please tell me the chords in the diagrams?*

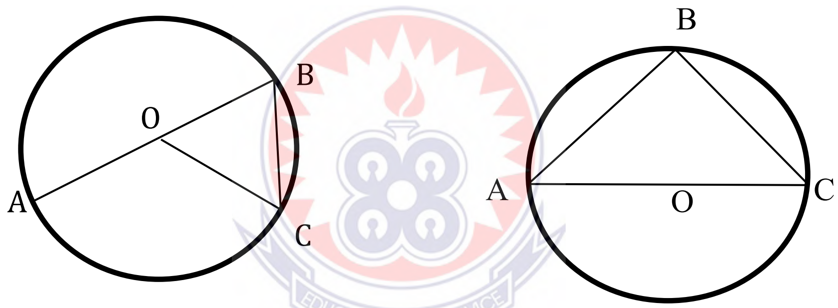


Diagram 1 Diagram 2

Esinam: *In diagram 2, BC is a chord from the diameter.*

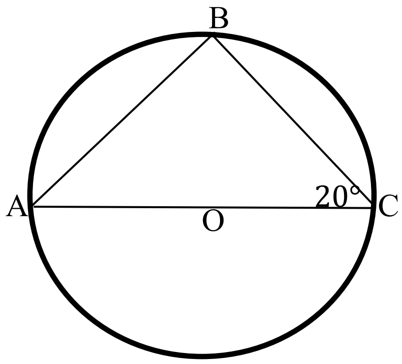
Researcher: *Can you also say that lines BC and AB in diagram 1 are also chord?*

Esinam: *(laughs and answer) Yes.*

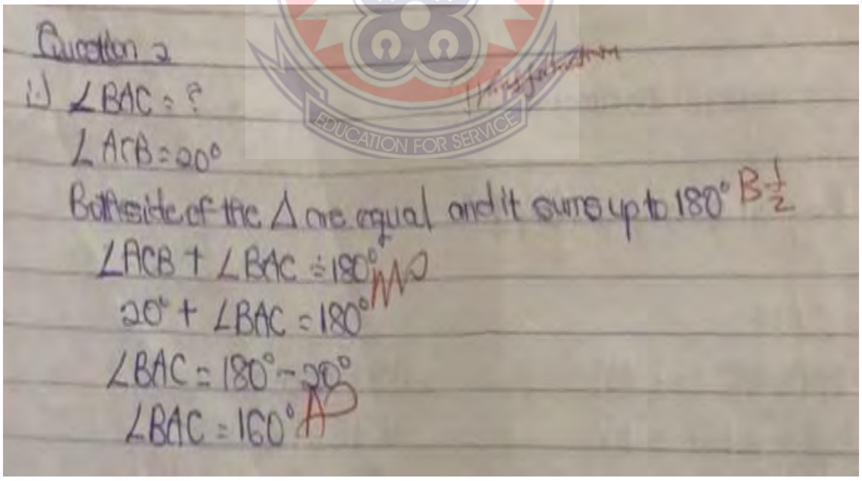
The interview responses showed that even though, she knows a chord touches the endpoints of a circle, she was not certain about the application to the question. The misconception noticed is that the student did not know that diameter is also a chord.

The questions in item 2 demanded the application of the knowledge of the theorem used in the previous question. Responses from the students indicated that most students could not identify that $\angle ABC = 90°$ in the diagram and therefore had the solution to be wrong. An interview with Ayorkor regarding item 2(i):

Researcher: *How did you get your solution in item 2(i)?*



Ayorkor: *Triangle ABC is an isosceles triangle, which means* $\angle ACB$ *and* $\angle BCA$ *are equal base angles. This implies that* $\angle BCA = \angle ACB$. *Therefore, knowing that the interior angle of a triangle adds up to* 180. *I added* $\angle ACB$ to$\angle BAC$ *and equated them to* 180°. *since one angle of the base has been given to be* 20°, $180° - 20° = 160°$.



Example of Ayorkor's response on item 2

Researcher: *But a triangle has three sides, so why did you add* $\angle BAC$ *to* $\angle ACB$ *and subtracted from* 180°?

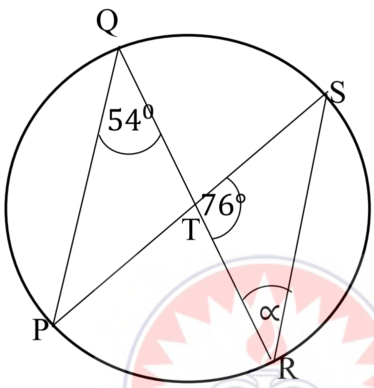
Ayorkor: *I didn't know what to do, but I remembered interior angles of a triangle add to* 180° *so I used it since the triangle is isosceles.*

Researcher: *why is the triangle isosceles?*

Ayorkor: *Because* $AB = BC$ *since they are both radii.*

This item needed the students to define the theorem that will be used to find the angle $\angle RST$ and also state the relationship between the angles $\angle RST$ and $\angle PQR$ and use it to calculate for a missing angle. Most of the students had the angle QRS to be wrong. In the following extract, Kuukua tries to explain her solution:

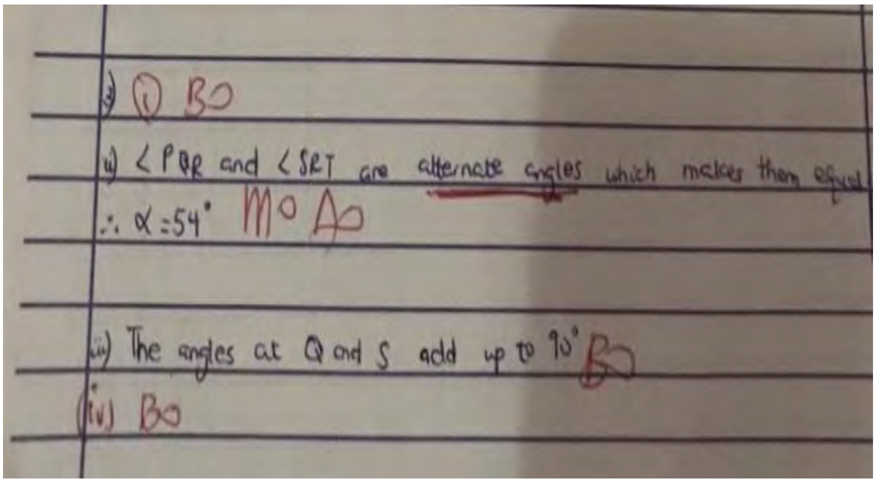
Researcher: *How did you get* $\angle QRS = 54°$ *in the diagram?*



Kuukua: $\angle QRS = 54°$ *because it's an alternating angle and alternating angles are equal.*

Researcher: *How did you get to know the angles are alternating?*

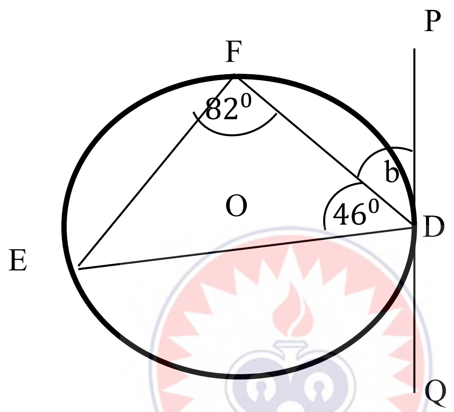
Kuukua: *erm since the diagram is forming a 'Z' symbol that's why they are alternating.*



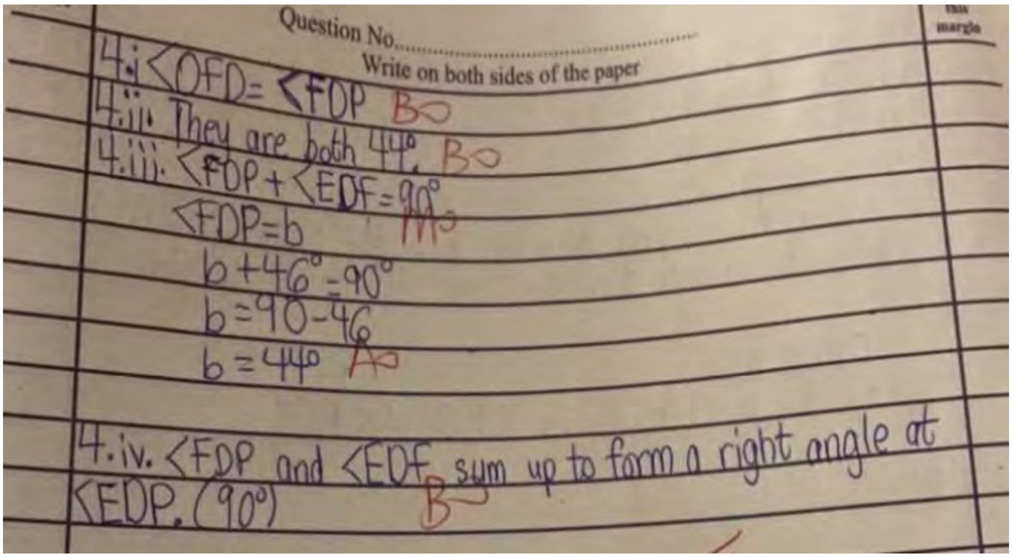
Example of Kuukua's response on item 3

In item 4, students were to identify the equal angles in the diagram and use the identified angles to solve for the missing angle. Some of the students could not recognize the angle to be angles formed between a chord and a tangent at the point of contact is equal to the angle formed in the alternate segment. They used angle formed between a tangent and a radius. This is what transpired between the researcher and a student in the interview:

Researcher: *Can you please explain how you calculated for ∠ PDF (b) in the diagram?*



Ewurama: *hmm, the circle is being attached by a tangent, which makes an angle that is* 90° *so you have to add both angles i.e.,* 46° *to angle b and equate to* 90°.



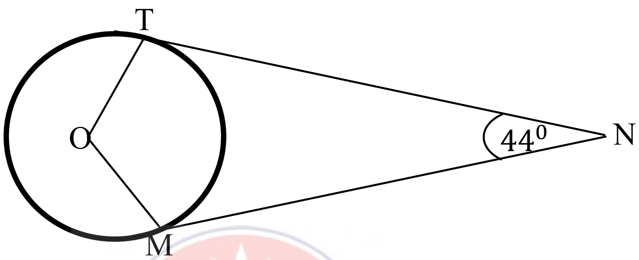
Example of Ewurama's response on item 4

Researcher: *How did you know the angle formed at D is* 90°?

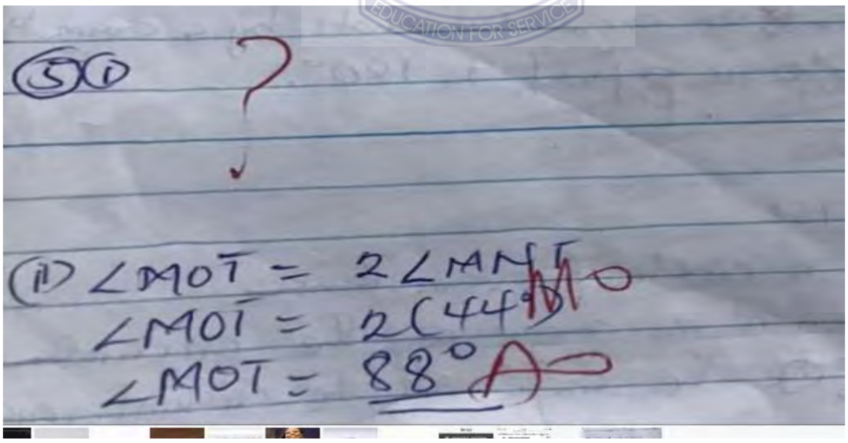
Ewurama: *From the diameter, it subtends a chord to the circumference at an angle. And I learnt that angle formed by a diameter and a tangent is* 90°.

An interview with Akuba regarding item 5 resulted in the following observations:

Researcher: *You stated in your work that* $\angle MOT = 2\angle MNT$. *Can you please explain to me why you responded that way?*



Akuba: *When we were learning the theorems of circles, I remember one theorem that stated that angle at the centre is twice the angle formed at the circumference. If you check the angles,* $\angle MOT$ *is at the centre, therefore twice* $\angle MNT$ *will give you* $\angle MOT$.



Example of Akuba's work on item 5